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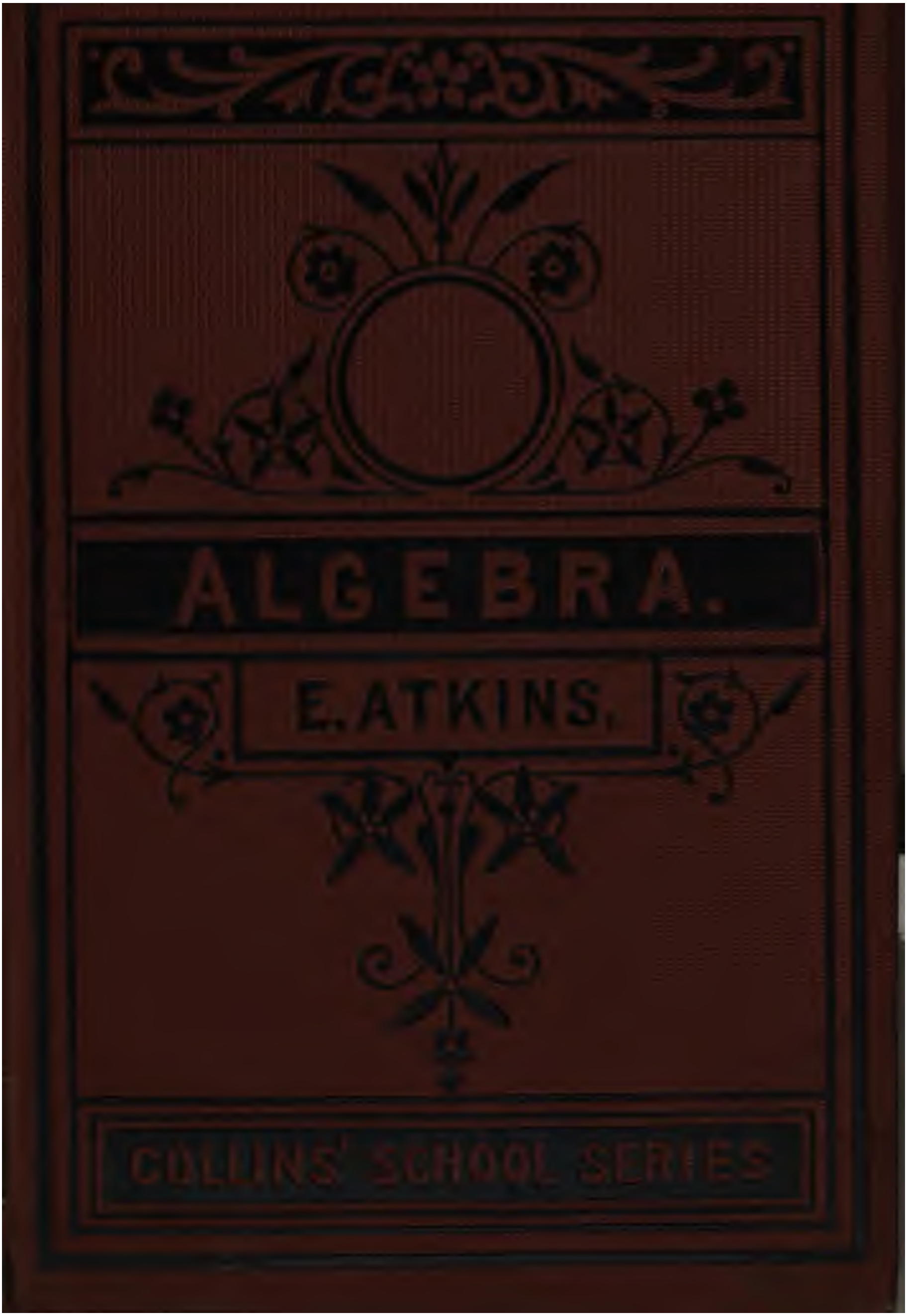

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ALGEBRA.

CHAPTER I.

ELEMENTARY PRINCIPLES.

1. **Algebra** treats of numbers, the numbers being represented by letters (symbols of *quantity*), affected with certain symbols of *quality*, and connected by symbols of *operation*.

It is easy to see that these symbols of quantity may be dealt with very much as we deal with concrete quantities in arithmetic. Thus, allowing the letter a to stand for the number of units in any quantity, and allowing also $2\,a$, $3\,a$, $4\,a$, &c., to stand respectively for *twice*, *thrice*, *four times*, &c., as large a quantity as the letter a, it at once follows that we may perform the operations of addition, subtraction, multiplication, and division upon these symbols exactly as we do in ordinary arithmetic upon concrete quantities. For instance, $4\,a$ and $6\,a$ make $10\,a$, $9\,a$ exceeds $5\,a$ by $4\,a$, $15\,a$ is 5 times $3\,a$, and $7\,a$ is contained 8 times in $56\,a$.

Neither is it necessary in these operations to *state*, or even to *know* the exact number of units for which any symbol of quantity stands, nor indeed the nature of these units; it is simply sufficient that it is a symbol of *quantity*. Thus, in the science of chemistry, we use a weight called a *crith;* and a person unacquainted with chemistry might not know whether a crith were a measure of length, weight, or capacity, or indeed whether it were a measure at all, yet he would at once allow that 6 criths and 5 criths are 11 criths, that twice 4 criths are 8 criths, &c.

The Signs $+$ and $-$ as Symbols of Operation.

2. In purely arithmetical operations, the signs $+$ and $-$ are respectively the signs of addition and subtraction. In this sense, too, they are used in algebra.

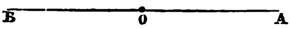
Thus, $a + b$ means that b is to be *added* to a, and $a - b$ means that b is to be *subtracted* from a.

Hence, as long as a and b represent ordinary arithmetical numbers, $a + b$ admits of easy interpretation, as also does $a - b$, when b is not greater than a. But when b is greater than a, the expression $a - b$ has no arithmetical meaning. By an extension, however, of the use of the signs $+$ and $-$, we are able to give such expressions an intelligible signification, whatever may be the quantities represented by a and b.

Positive and Negative Quantities.—The Signs $+$ and $-$ as Symbols of Affection or Quality.

3. Def.—A **positive** quantity is one which is affected with a $+$ sign, and a **negative** quantity is one which is affected with a $-$ sign.

Let BA be a straight line, and O a point in the line; and suppose a person, starting from O, to walk a miles in the direction OA. Suppose also another person, starting from the same or any other point in BA to walk a miles in the direction OB. These persons will thus walk a miles each in *exactly opposite* directions. Now, we call one of these directions *positive* (it matters not which) and the other *negative*. Let us take the direction OA as *positive*. We then have the first person walking a miles in a *positive* direction, and the second walking a miles in a *negative* direction. We represent these distances algebraically by $+a$ and $-a$ respectively.

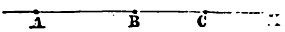


It will therefore be seen that the signs $+$ and $-$ have no effect upon the magnitudes of quantities, but that they express the *quality* or *affection* of the quantities before which they stand.

Again, suppose a person in business to get a profit of £6, while another suffers a loss of £6. We may express these facts algebraically in two ways. We may consider *gain* as positive, and *loss* as negative gain, and say that the former has gained + 6 pounds, while the latter has gained − 6 pounds. Or we may consider *loss* as positive, and *gain* as negative loss, and say that the former has lost − 6 pounds, while the latter has lost + 6 pounds. We hence see that the gain of + 6 is equal to a loss of − 6, and that a gain of − 6 is equal to a loss of + 6.

The Sum of Algebraical Quantities.

4. Let a distance AB be measured to the *right* along the line AX. And let a further distance BC be measured from B in the *same* direction. By the *sum* of these lines we mean the resulting distance of the point C from the original point A, that is to say, the distance AC.



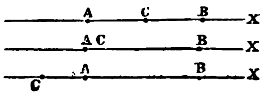
(It may be remarked that we *add* the line BC to the line AB by measuring BC in its *own proper direction* from the extremity B of AB. It is hardly necessary to remind the student that both lines are in the same straight line AX.)

Let us represent the distances AB and BC by + *a* and + *b* respectively; then the *algebraical sum* of the lines will be represented writing these quantities side by side, each with its *own proper sign* of affection.

Thus the sum of the distances AB and BC is expressed by + *a* + *b*, or, as it is usual to omit the + sign of a positive quantity when the quantity stands alone or at the head of an algebraical expression, the sum of AB and BC is expressed by *a* + *b*.

Hence, the interpretation of *a* + *b* is that it represents the distance AC.

Again, taking as above + *a* to represent the distance AB along the straight line AX, and measured to the *right*, let a distance BC be measured from B in the same straight line AX, but this time to the *left*.



Let the latter distance be represented by $-b$.

Then, on the principle above, AC is the *sum* of these distances, and this *sum* is represented algebraically by $+a-b$ or $a-b$.

(It will be seen that the distance BC is again measured from B in its *own proper direction*, and that the resultant distance AC of the point C is again the *sum* of the line AB and BC.)

There will evidently be three cases, viz. :—

1. When the distance BC is less in magnitude than AB, in which case the point C is on the right of A, and the distance AC is *positive*.

2. When the distance BC is *equal* in magnitude to AB, in which case the point C coincides with A, and the distance AC is *zero*.

3. When the distance BC is *greater* in magnitude than AB, the point C being then on the left of A, and the distance is *negative*.

Now, $a-b$ in all these cases represents the distance AC. It therefore admits of intelligible interpretation whether b be less than, equal to, or greater than a.

And, since the distance AC is obtained in the first two cases by subtracting the distance BC from that of AB, and in the second case by subtracting as far as AB will allow of subtraction, and measuring the remainder to be subtracted in an *opposite* direction, it follows that—

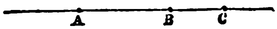
The sign $-$, which, standing before a letter, is a symbol of quality, becomes at once a symbol of subtraction in all cases when the quantity in question is placed immediately after any other given quantity with its proper sign of affection.

Hence also we may conclude that the addition of a *negative* quantity is equivalent to the subtraction of the corresponding *positive* quantity.

5. We may prove in a similar way that—

The subtraction of a *negative* quantity is equivalent to the addition of the corresponding *positive* quantity.

Let, as before, $+a$ represent the distance AB, measured from the point A to the *right*, and let it be required to *subtract* from $+a$ the distance represented by $-b$.



Now, in the last article, we *added* a distance to a given distance AB, by measuring the second distance in its *own direction* from the extremity B. We shall therefore be consistent if we *subtract* a given distance $-b$ by measuring this distance in a direction *exactly opposite* to its own direction, from the same extremity B.

Now, the direction of $-b$ is to the *left*. If, therefore, we measure a distance BC to the *right*, equal in magnitude to the distance to be subtracted, we obtain a distance AC which is correctly represented by $a - (-b)$. But AC is also correctly represented by $a + b$, and hence it follows that $a - (-b) = a + b$.

We may apply the above principle to all magnitudes which admit of continuous and indefinite extension; as, for instance, to forces which pull and push, attract and repel; to time past and time to come, to temperatures above zero and below zero, to money due and money owed, to distance up and distance down, &c., in all which cases, having represented one by a quantity affected with a + sign, we may represent the other by a quantity affected with a − sign.

6. *In expressing the sum of a number of quantities, the order of the terms is immaterial.*

We will take, as our illustration, a body subject to various alterations of temperature, and we will suppose the temperature of the body, before the changes in question, to be *zero* or 0°. Let the temperature now undergo the following changes—viz., a rise of $a°$, a fall of $b°$, a fall of $c°$, and, lastly, a rise of $d°$. Let us consider a *rise* as positive, and therefore a *fall* as negative. We may then represent these changes respectively by $+a$, $-b$, $-c$, $+d$.

And it is further evident that the resulting temperature will be represented by the *sum* of these quantities, which, as previously written, will be $a - b - c + d$.

But again, it is plain that the resulting temperature of the body will not be affected if these changes of temperature take place in the reverse order, or in any other order. Thus, suppose the temperature first falls $c°$, then rises $a°$, then rises $d°$, and, lastly, falls $b°$, it is evident that the final temperature will be the same as before. And the *sum* of the quantities $-c$, $+a$, $+d$, $-b$, represents this final temperature.

Now, expressing the sum of these quantities by writing them (Art. 4) side by side with their proper signs of affection, and in the order in which they stand, we have $-c+a+d-b$ for the *sum.*

It therefore follows, since we might have chosen any other order of these terms with a similar result, that the sum of any number of quantities, $+a$, $-b$, $-c$, $+d$, may be expressed by writing the terms side by side with their proper signs of affection *in any order whatever.*

Nevertheless, for convenience, and for other reasons, we write the terms *generally* in *alphabetical order*, or we arrange them according to the *power* (see Art. 16) of some particular letter.

7. We may sum up the results and remarks of the last four articles as follows :—

1. *Positive* and *negative* are used in exactly opposite senses.

2. The sign $+$ before an algebraical quantity *affirms* the quality of the quantity as represented without the sign in question.

Thus, $+(+a) = +a$, and $+(-b) = -b$.

3. The sign $-$ before an algebraical quantity *reverses* the quality of the quantity as represented without the sign in question.

Thus, $-(+a) = -a$, and $-(-b) = +b$.

4. The algebraical subtraction of a quantity is the same as the addition of the quantity with the sign of affection *reversed.*

5. The algebraical addition of quantities is expressed by writing the quantities down side by side with their proper signs of affection. And they *may* be written down in any order, though we generally write them in *alphabetical* order, or arranged according to the power (Art. 16) of some letter.

Brackets.

8. **Brackets**—( ), { }, [ ]—are used, for the most part, whenever we wish to consider an algebraical expression containing more than one term *as a whole.*

Thus, if we wish to express that the quantity $3a + 7b$ is to be added as a whole to $4a$, we write—

$$4a + (3a + 7b),$$

and, while inclosed within brackets, we think and speak of $3a + 7b$ as *one quantity*.

Again, if we wish to express that $b - c$ is to be subtracted from a, we write—

$$a - (b - c).$$

Let us consider what is the result of subtracting $(b - c)$ from a. We may evidently, if we please, subtract b first, then afterwards $- c$ from the quantity so obtained, without affecting this result.

Now, we know by Art. 7 (4.) that this is equivalent to *adding* the quantities $- b$ and $+ c$ successively.

Now, the sum of a, $- b$, $+ c$, is $a - b + c$.

We have therefore $a - (b - c) = a - b + c$.

We observe that the sign of b *within* the brackets is $+$, and that of c is $-$, whereas, in our final result, these signs are both reversed. And we hence arrive at the following important principle:—

When a *minus* sign stands before a bracket, its effect on removing the brackets is to *reverse* the sign of affection of every term *within*.

And it is evident that we may, by a similar course of reasoning, arrive at a principle equally important, viz.:—

When a *plus* sign stands before a bracket, its effect on removing the bracket is to *affirm* the sign of affection of every term within.

We shall show, in Art. 9, the use of brackets in expressing the product or quotient of quantities.

Though, as stated above, brackets are, for the most part, used to group together as a whole a number of quantities, they are sometimes used to inclose single terms. Thus, in Art. 5, we have the expression $a - (- b)$. Now, the brackets are used here to express that the *negative quantity* is to be subtracted as a *negative quantity*. And, in the same way, the expression $a + (- b)$ indicates that the *negative quantity* $(- b)$ is to be *added* to the quantity a. When one pair only is required we generally use the brackets ( ); if, however, a quantity already in brackets is to be inclosed in a second pair, we use $\{\ \}$, as in the expression—

$$3a - \Big\{6b + (4c - d)\Big\}.$$

If a third pair be required we use the brackets $[\ ]$, and finally, we sometimes find it convenient to group a number of terms by means of a *vinculum*, thus—

$$4x - \Big[6x - \big\{5y + (3z - \overline{7x - y})\big\} + 2y\Big].$$

It must be remembered that the *vinculum* has in an expression exactly the same force as brackets.

9. We shall, in this Article, show how to find the value of a few algebraical expressions, as illustrations of the foregoing principles :—

Ex. 1.—If $a = 1$, $b = 2$, $c = 4$, find the value of $3a + 5b + 7c$.

We have only to substitute the value of the letters in the given expression, putting a sign of multiplication to avoid ambiguity.

Thus we have—

$$\begin{aligned} 3a + 5b + 7c &= 3 \times 1 + 5 \times 2 + 7 \times 4. \\ &= 3 + 10 + 28 = 41. \end{aligned}$$

Ex. 2.—If $x = 5$, $y = 2$, $z = 6$, find the value of $4x + 3y - 9z$.

$$\begin{aligned} \text{Here, } 4x + 3y - 9z &= 4 \times 5 + 3 \times 2 - 9 \times 6 \\ &= 20 + 6 - 54 \\ &= 26 - 54. \end{aligned}$$

The negative quantity is here the larger, and exceeds the positive quantity by 28, and hence (Art. 4 (3),) the result will be *negative.*

We therefore have—

$$4x + 3y - 9z = -28.$$

Ex. 3.—If $x = 1$, $y = 3$, $z = 0$, find the value of—

$$3x - \Big\{2y + (6z - \overline{5x - y})\Big\}.$$

The given expression—

$$\begin{aligned} &= 3 \times 1 - \Big\{2 \times 3 + (6 \times 0 - \overline{5 \times 1 - 3})\Big\} \\ &= 3 - \Big\{6 + (0 - \overline{5 - 3})\Big\} \end{aligned}$$

$$= 3 - \{6 + (0 - 2)\}.$$
$$= 3 - (6 - 2) = 3 - 4 = -1.$$

It may be advisable to simplify expressions of this kind before substituting the given value of the letters. The method of doing this, however, is deferred till we come to Chapter II.

Ex. I.

If $a = 1$, $b = 2$, $c = 3$, $d = 0$, $e = 4$, find the value of the following :—

1. $4a + 2b$, $3b + 7c$, $6a + 4d$, $4c - 7e$.
2. $a + b + c$, $a - b + c$, $b + c - a$, $a + b - c$.
3. $3a + 7b - 4c$, $2a + 7d + 3c$, $7a - 10b + 2c$.
4. $6a - (2b - 3c)$, $7b + (2a - 4d)$.
5. $3c + (6a - \overline{7b + c})$, $2b - \{7c + (4d - b)\}$.
6. $\{7b - (2d - c)\} - (4b - c + e)$.

If $x = 2$, $y = 3$, $z = 4$, find the sum and difference of the following expressions :—

7. $3x - 4y$ and $3y - 4x$.
8. $7(x - y)$ and $4(y - z)$.
9. $3x - (7y + 4z)$ and $8y + 5z - 3x$.
10. $12y - \overline{6y + 4z}$ and $12y + \overline{6y + 4z}$.
11. $x - y - z$ and $x - (y - z)$.
12. $x - (-3y)$ and $y - \{6x - (-3z)\}$.

Product of Two or more Quantities.

10. The product of two or more quantities may be expressed in several ways.

Thus, the product of a, b, c may be written as follows :—

1. abc, by placing the letters side by side without *any sign* between them.
2. $a \times b \times c$, by placing between them the sign ×.
3. $a \, . \, b \, . \, c$, by placing a dot between them.

4. $(a)\,(b)\,(c)$, by inclosing the quantities in brackets, writing them without any sign between the brackets.

When, however, the quantities are negative, or either of them consists of more than one term, it is *best* to inclose such quantities in brackets, and, in most cases, it is *necessary* to do so.

Thus, the product of a, $-\,b$, c would not be correctly expressed by $a - bc$ (for this means, that the product of b and c is to be subtracted from a), but must be written $a\,(-\,b)\,c$. Again, the product of $2\,a + 3\,b$ and $a + 5\,b$ cannot be written $2\,a + 3\,ba + 5\,b$, as this expression means that three times the product of $b \times a$ is to be added to $2\,a$, and then $5\,b$ to the result. The product is correctly expressed thus—

$$(2\,a + 3\,b)\,(a + 5\,b).$$

(The student cannot be too strongly cautioned against leaving out brackets in cases of this kind.)

The Order of the Letters.

11. It is evident that a times $b = b$ times a; for, if we arrange a rows of b things so as to have a horizontal rows and b vertical columns, we may either consider the number of *rows* or the number of columns. In the former case we have a times b things, and, in the latter, b times a things.

We may therefore write the letters whose product we wish to express in any order.

Thus, the product of a, b, c may be written in either of the following ways:—

$$abc,\ acb,\ bac,\ cba,\ bca,\ cab.$$

For convenience, however, and for reasons which the student will see as he proceeds with the subject, we write them in the order of the alphabet, unless there happen to be special reasons to the contrary.

Rule of Signs in Multiplication.

12. It was shown (Art. 3) that a *minus* sign does not affect the magnitude of a quantity, but simply its *affection* or *quality*. It therefore follows that the product of $+a$ and $-b$ will be the same in magnitude as that of $+a$ and $+b$, or of $a \times b$—that is, will be equal in magnitude to ab. But it is evident that the quality of the product will be the same as that of $-b$, for it is a times *a negative* quantity.

We therefore have $+a(-b) = -ab$.

So also, the product of $-a$ and $+b$, will have the same magnitude as that of $+a$ and $+b$—that is, its magnitude will be ab. But since, to take a geometrical illustration, a line of length, b, drawn *negatively once, negatively twice*, &c., must give a negative result, the quality of the product in question must be negative.

We therefore have $(-a)(+b) = -ab$.

Again, the product of $-a$ and $-b$ will be equal in magnitude to ab, and its quality will be evidently the *reverse* of the quality of the product of $-a$ and $+b$, and the quality of the latter product is above shown to be *negative*. The product of $-a \times -b$ will be therefore positive.

Hence we have $(-a)(-b) = +ab$.

Collecting these results, and remembering that $(+a)(+b) = +ab$, we have—

$$\begin{aligned}(+a)(+b) &= +ab.\\(+a)(-b) &= -ab.\\(-a)(+b) &= -ab.\\(-a)(-b) &= +ab.\end{aligned}$$

We have then the following rule of signs in multiplication :—

RULE.—Like signs give $+$, and unlike signs give $-$.

Quotient of Two Quantities.

13. The quotient of two quantities is expressed in either of the two following ways :—

1. By placing the divisor under the dividend, separated by a line.

Thus, the quotient of a by $b = \frac{a}{b}$.

2. By placing the divisor after the dividend with a sign $\div$ between them; thus, $a \div b$.

When either of the quantities is negative, it is better, if the second be used, to inclose the negative quantity, if the divisor, in brackets.

Thus, the quotient of a by $-b$ may be written $a \div -b$, but it is better written $a \div (-b)$.

And when either of the quantities contains more than one term, it must always be inclosed in brackets when expressed by the second method.

Thus, the quotient of $a + 2b$ by $2c = (a + 2b) \div 2c$, not $a + 2b \div 2c$, for this would mean that the quotient of $2b$ by $2c$ is to be added to a.

Rule of Signs in Division.

It is evident from the last article that—

$$\begin{aligned} (+ab) \div (+a) &= +b. \\ (-ab) \div (+a) &= -b. \\ (-ab) \div (-a) &= +b. \\ (+ab) \div (-a) &= -b. \end{aligned}$$

We have therefore the following rule :—

Rule.—In division, as in multiplication, like signs give +, and unlike signs give −.

Coefficients.

14. When a quantity can be broken up into two factors, each of those factors is called a **coefficient** of the other.

Thus, taking the quantity $4abc$, we see that 4 is the coefficient of abc, $4b$ that of ac, $4c$ that of ab, $4bc$ that of a, &c. We call 4 a *numerical* coefficient; but when the coefficient contains a letter or letters we call it a *literal* coefficient. We often speak of the numerical coefficient of a quantity as *the coefficient* of the quantity. It is, moreover, unusual to write down unity as a numerical coefficient, and so, when an algebraical quantity has no expressed numerical coefficient, we may, conversely, consider *unity* as such.

15. *Like quantities* are generally defined as those which differ only in their *numerical* coefficients. Thus, $9\,x$, $3\,y$, $4\,xyz$, $12\,ab$ are respectively *like* to the quantities $10\,x$, $5\,y$, $6\,xyz$, $4\,ab$, while $3\,x^2$, $6\,xy$, $7\,b^2$ are called *unlike*.

It is sometimes convenient, however, to *consider as like quantities* those which may contain perhaps one *common* letter only, though containing other letters which are not common. Thus, for the purposes of addition and subtraction, we may consider $3\,ax$ and $4\,bx$ as *like quantities*, having $3\,a$ and $4\,b$ respectively as their coefficients. Now, their sum, since they are respectively equivalent to $3\,a$ and $4\,b$ times x, will be equivalent to $(3\,a + 4\,b)$ times x, and may then be written $(3\,a + 4\,b)\,x$.

Hence we learn that addition and subtraction of quantities having any common factor may, considering the uncommon factor as coefficient, be expressed according to the following rule :—

Add together the coefficients of the common factor, and take the sum as a new coefficient of the common factor.

And a similar rule will apply to the subtraction of such quantities.

Thus, we have the sum of $4\,axy$, $-\,2\,bcy$, $3\,dey$,
$= (4\,ax - 2\,bc + 3\,de)\,y$.

Powers.

16. We have seen (Art. 10) that abc means that a is to be multiplied by b, and the product by c. And so, if we wish to express that a is to be multiplied by a, and this product again by a, we might write the expression thus: aaa. Instead of this, however, we usually place a small figure at the top and on the right hand of the letter a, to indicate how many times the letter appears as a factor. In this case, therefore, we write a^3. We call quantities of this form **powers** of the letter in question; and so, remembering that a may be considered as a^1 on this principle, we have :—

a, the first power of a; a^2, the *second* power of a; a^3, the *third* power of a, &c. The figure written at the right hand at the top of the letter is called the **index** or **exponent**, and it is usual to call a^2, a^3 respectively the *square* and *cube* of a, from the fact they express respectively the *area* of a square whose side is a, and the *volume* of cube whose edge is a.

17. The **square root** of a quantity is that quantity which, when raised to the *second power* or *squared*, will give the original quantity.

It is generally written $\sqrt{\ }$. Thus, $\sqrt{16} = 4$, $\sqrt{144} = 12$.

The **cube root** of a quantity is that quantity which, when raised to the *third power* or *cubed*, will give the original quantity.

It is generally written $\sqrt[3]{\ }$. Thus, $\sqrt[3]{8} = 2$, $\sqrt[3]{27} = 3$, $\sqrt[3]{1728} = 12$. And so the fourth, fifth, &c., roots are indicated by the symbols $\sqrt[4]{\ }$, $\sqrt[5]{\ }$, &c., respectively.

18. The **dimensions** of an algebraical quantity are the sum of the indices or exponents of the *literal* factors.

Thus, the dimensions of $3\,a^2b^3c^4 = 2 + 3 + 4 = 9$, and the dimensions of $abx^2 = 1 + 1 + 2 = 4$.

19. A **homogeneous** expression is one in which the dimensions of every term are the same.

Thus, $a^3 + 3\,a^2b + 3\,ab^2 + b^3$ is *homogeneous*,
Whereas, $5\,a + 3\,ab + 7\,a^2bc$ is *not homogeneous*.

Ex. II.

If $a = 2,\ b = 3,\ c = 0,\ d = 1$,

Find the values of—

1. $6a^2 + 3\,b^2 - 5\,c^2$; $ab + ac + bc$, $bc + bd + cd$.
2. $a^3 + 3\,a^2b + 3\,ab^2 + b^3$; $a^3 + b^3 + c^3 - 3\,abc$.
3. $(3\,a + 7\,b)\,(4\,a - 9\,b)$; $(a^2 + b^2)\,(a + b)\,(a - b)$.
4. $6\,\left\{2\,a^3 - 4\,\left(2\,b^3 - \overline{2\,c^3 - d^3}\right)\right\}$; $a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2$.

If $a = 1,\ b = -\,2,\ c = -\,3,\ d = 0,\ e = 4$,

Find the values of—

5. $(a + b + c + d + e)^2$; $(a^2 + 2\,ab + b^2 - c^2) \div (a + b + c)$.
6. $\dfrac{a^4 - d^4}{a^3 + a^2d + ad^2 + d^3}$; $\dfrac{c^3 + 3\,c^2d + 3\,cd^2 + d^3}{b^3 - 3\,b^2c + 3\,bc^2 - c^3}$.
7. $\dfrac{\sqrt{c^2 + e^2} - \sqrt{5\,a^2 + b^2}}{ab + ac + bc}$; $\sqrt[3]{c^3 + 3\,c^2 + 3\,c + 1}$.

8. $3(4b + 5c)^3 + 4(c + e)^2$; *abcde.*

If $x = 3,\ y = 4,\ z = 0$,

Find the value of—

9. $(3x - \sqrt{x^2 + y^2})^2 (2x + \sqrt[3]{x^2 + y^2 + 2})$.

10. $\{5x^2 + 2(y + 2)^2\}\{5x^2 - 2(y + z)^2\}$.

11. $x^y + y^x + z^x$.

12. $(x^3 - y^3) \div \sqrt[3]{\{3x^3 + 3(3x^2 + 3xy + y^2)y\}}$.

If $a = 7,\ b = 4,\ c = 9$, then—

13. $(a - \sqrt{c})(a^2 - b) + \sqrt{a^2 - b - c}(\sqrt{b} - \sqrt{c})^2 = 186$.

14. $a^2(c - \sqrt{b^2 - a}) + b^2(a - \sqrt{a - b^2 + c}) = 406$.

15. $\dfrac{3a - b}{2a - c} + \dfrac{5b + a}{b - c} + \dfrac{7c - a}{a + 2c} = \dfrac{6}{25}$.

16. $\dfrac{a^3}{(a-b)(a-c)} + \dfrac{b^3}{(b-c)(b-a)} + \dfrac{c^3}{(c-a)(c-b)} = 20$.

17. $\dfrac{a^2b - ab^2 + b^2c - bc^2 + ac^2 - a^2c}{(a - c)(a - b)} = -5$.

18. $\dfrac{a + 5c}{b} - \left(\dfrac{a}{7} - \dfrac{b - 10}{c}\right) - 5a = -23\dfrac{2}{3}$.

If $x = 10,\ y = 11,\ z = 12$, then—

19. $(x-y)(y-z) + (y-z)(z-x) + (z-x)(x-y) = -3$.

20. $\left(\dfrac{x+y}{x-y} + \dfrac{x^2+y^2}{x^2-y^2}\right) \div \left(\dfrac{x-y}{x+y} - \dfrac{x^3-y^3}{x^3+y^3}\right) = -334\dfrac{1}{110}$.

21. $\{x^2 - (y + z)^2\} \cdot \dfrac{x + y - z}{x + y + z} = -117$.

22. $\{(x + y)^2 - (x - y)^2 - 4y^2\} \div (x - y) = 44$.

23. $\left(x + \dfrac{1}{y + \dfrac{1}{z}}\right)(y^2z^2 - 1) = 175802$.

24. $$\frac{\dfrac{1}{x+y}+\dfrac{1}{z}}{\dfrac{1}{x+y}-\dfrac{1}{z}}(z^2-x^2-y^2-2\,xy)=1089.$$

25. $$\frac{y\,(x+z)-2\,z}{xy-1}\cdot\sqrt{\frac{z\,(y+1)}{y\,(z-1)}}=2\,\frac{2}{11}.$$

26. $$\left(\frac{2\,x}{y-1}\right)^2-\left(\frac{y}{z-1}\right)^2+\left(\frac{z}{y+1}\right)^2=4.$$

CHAPTER II.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION.

Addition.

20. Rule.—Arrange the terms of the given quantities so that *like* quantities may be under each other; add separately the positive and negative coefficients of each column; take the difference and prefix the sign of the *greater*, and annex the common letter.

(When the coefficients are all positive or all negative, we, of course, simply add them together and prefix the common sign for the coefficient of the sum.)

Ex. 1.

$$\begin{array}{lrrr}
 & 3\,a & +\,5\,b & -\,3\,c \\
 & 2\,a & -\,7\,b & +\,4\,c \\
 & 5\,a & +\quad b & -\,2\,c \\
 & 4\,a & -\,3\,b & +\,8\,c \\
\hline
\text{Ans.} & 14\,a & -\,4\,b & +\,7\,c
\end{array}$$

Ex. 2.

$$\begin{array}{lrrrr}
 & -\,3\,a & +\,7\,b & +\quad c & -\,4d \\
 & 2\,a & -\,2\,b & +\,5\,c & +\,3d \\
 & -\,7\,a & -\,3\,b & +\,2\,c & -\,6d \\
 & 2\,a & +\,5\,b & -\,8\,c & +\,6d \\
\hline
\text{Ans.} & -\,6\,a & +\,7\,b & & -\quad d
\end{array}$$

Ex. 3. Add together $5\,x^2-3\,y^2+3\,y$, $6\,y^2+7\,xy-4\,x$, $4\,xy+6\,x-5$, $-2\,x^2-3\,xy+2$.

Arranging like quantities in each expression under each other, we have :—

$$\begin{array}{lrrrrrr}
 & 5\,x^2 & & -\,3\,y^2 & & +\,3\,y & \\
 & & 7\,xy & +\,6\,y^2 & -\,4\,x & & \\
 & & 4\,xy & & +\,6\,x & & -\,5 \\
 & -\,2\,x^2 & -\,3\,xy & & & & +\,2 \\
\hline
\textit{Ans.} & 3\,x^2 & +\,8\,xy & +\,3\,y^2 & +\,2\,x & +\,3\,y & -\,3
\end{array}$$

Ex. III.

Add together—

1. $3a - 2b, 4a + 7b, 2a + 3b, a - 5b.$

2. $9a^2 + 7b^2, -3a^2 + 4b^2, a^2 + b^2, 4a^2 - 12b^2.$

3. $a + b + c, 3a + 2b + 3c, -4a + 7b - c, 2b + 5c.$

4. $x - y - z, y - x - z, z - x - y, x + y + z.$

5. $3a^2 - 4ab + 6b^2, 7ab - a^2 - b^2, 2a^2 - 3ab - 4b^2, 4a^2 + ab - b^2.$

6. $2x^4 - 7x^2 + 3, -4x^3 + 6x^2 - 2x + 7, x^4 - 2x^3 - 4x, 6x^3 - 9x - 12.$

7. $2a^2 + 7ab + 3b^2 - 6a - 5b - 2, a^2 + 3a - 2b + 9, 9ab - 2a - 3b + 4, -3a^2 - 12ab - 3b^2 + 5a + 10b - 15.$

8. $x^3 - x^2y - x^2z + xy^2 - xyz + xz^2, x^2y - xy^2 - xyz + y^3 - y^2z + yz^2, x^2z - xyz - xz^2 + y^2z - yz^2 + z^3.$

9. $x^4 + x^2y^2 + x^3y, -x^3y - x^2y^2 - xy^3, y^4 + xy^3 + x^2y^2.$

10. $a^3 + ab^2 + ac^2 + 2a^2b - 2a^2c - 2abc, a^2b + b^3 + bc^2 + 2ab^2 - 2abc - 2b^2c, a^2c + b^2c + c^3 + 2abc - 2ac^2 - 2bc^2.$

11. $x^4 - xy^3 + xz^3 - 3x^3y + 3x^3z, 3x^2y^2 + 3x^2z^2 + 3xy^2z - 3xyz^2 - 6x^2yz, y^4 - x^3y - yz^3 + 3x^2y^2 - 3x^2yz, -3xy^3 - 3xyz^2 - 3y^3z + 3y^2z^2 + 6xy^2z, z^4 + x^3z - y^3z - 3x^2yz + 3x^2z^2, 3xy^2z + 3xz^3 + 3y^2z^2 - 3yz^3 - 6xyz^2.$

12. $a^4 - a^3b + 3a^2c^2 + ab^2c - 3abc^2 - b^3c, a^3b - a^3c + 3abc^2 - 3ac^3 - b^2c^2 + b^3c, a^3c - a^3d + 3ac^3 - 3ac^2d + b^2c^2 - b^2cd, -a^4 + a^3d - 3a^2c^2 + 3ac^2d - ab^2c + b^2cd.$

Subtraction.

21. We have seen, Art. 7 (4.), that the subtraction of a quantity is equivalent to the addition of the same quantity with its sign of affection reversed. We therefore have the following rule:—

RULE.—Change the sign of each term of the subtrahend, and proceed as in addition.

Ex. 1.

$$\begin{array}{lrrr} & 5x & +\,6y & -\,3z \\ & 2x & -\,2y & +\,2z \\ \hline \text{Ans.} & 3x & +\,8y & -\,5z. \end{array}$$

Ex. 2.

$$\begin{array}{lrrr} & 6a^2 & -\,3ab & +\,4b^2 \\ & -\,2a^2 & -\,3ab & -\,2b^2 \\ \hline \text{Ans.} & 8a^2 & & +\,6b^2. \end{array}$$

Ex. 3.

$$\begin{array}{rrrr}
2x^3 & & +\ 6xy^2 & +\ 3y^3 \\
x^3 & +\ 3x^2y & +\ 3xy^2 & +\ 2y^3 \\
\hline
x^3 & -\ 3x^2y & +\ 3xy^2 & +\ \ y^3
\end{array}$$

Ex. IV.

1. From $6a + 7b + 3c$ take $2a + 5b - 2c$.

2. From $2x - 3y - 8z$ take $6x - 5y - 2z$.

3. Take $5a^2 + 3ab + 4b^2 + 3a + 7b + 8$ from $6a^2 + 3b^2 - 2a$.

4. Take $6a^4 + 8a^2x^2 + x^4$ from $8a^4 + 6a^2x^2 + 2x^4$.

5. Subtract the sum of the quantities $a^4 + 2a^2b^2 + b^4$, $a^4 - 2a^2b^2 + b^4$ from $6a^4 + 8a^2b^2 + 6b^4$.

6. From $x^3 + y^3 + z^3 - 3xyz$ take $4x^3 + y^3 + 4z^3 + 3x^2z + 3xz^2 - 3xyz$.

7. From $3x^4 + 3ax^3 - 9a^2x^2 + a^3x - a^4$ take $2x^4 + 4ax^3 + 4a^3x + a^4$.

8. Take $a^3 - 5a^2b + 7ab^2 - 2b^3$ from the sum of $2a^3 - 9a^2b + 11ab^2 - 3b^3$ and $b^3 - 4ab^2 + 4a^2b - a^3$.

9. Subtract $a + b + c + d$ from $e + f + g + h$.

10. Take $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ from $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$, and subtract the result from their sum.

11. Add together the given quantities in the last example, and subtract the result from $3x^4 + 10x^2y^2 + 3y^4$.

12. Take $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ from $2a^2 + 2b^2 + 4ab - c^2$.

Brackets—*continued.*

22. It was shown, in Art. 8, that, when a quantity inclosed in brackets is to be added, we may remove the sign (+) of addition and the brackets without changing the sign of the terms within the brackets. On the contrary, when the quantity in brackets is to be subtracted, or has the sign *minus* before it, we must change the sign of every term within the brackets on removing the brackets and the sign of subtraction.

We shall now see how to simplify expressions involving brackets connected by the signs of addition and subtraction:—

Ex. 1. Simplify $(3a + 5b) - (6b - 2c) + (-2a + b - 3c)$.

The given expression $= 3a + 5b - 6b + 2c - 2a + b - 3c$, or adding together the like quantities,

$$= a - c.$$

Ex. 2. Reduce to its simplest form—

$$a - (b - c) + \left\{b + (a - c)\right\} - \left\{(a - b) - c\right\}.$$

(When a pair of brackets is inclosed within another pair, it is convenient to remove the inner one first.)

Hence the given expression—

$$= a - b + c + \left\{b + a - c\right\} - \left\{a - b - c\right\}$$

$$= a - b + c + b + a - c - a + b + c = a + b + c.$$

Ex. 3. Simplify the expression $\frac{3x}{4} - \frac{x - 7}{2} + \frac{6x - 9}{8}$.

The line separating the numerator and denominator of a fraction is a species of vinculum, since it serves to show that the *whole* numerator is to be divided by the *whole* denominator. Hence, on breaking up the two latter fractions into fractions having one term only in the numerator, we have—

$$\frac{3x}{4} - \frac{x - 7}{2} + \frac{6x - 9}{8} = \frac{3}{4}x - \frac{1}{2}x + \frac{7}{2} + \frac{6}{8}x - \frac{9}{8} = x + \frac{19}{8}.$$

23. As it is often necessary to inclose quantities within brackets, we shall now show how this is done.

The following rule needs no explanation:—

Rule.—When a number of terms is inclosed within brackets, if the sign placed before the brackets be +, the terms must be written down with their signs of affection *unchanged;* but, if the sign placed before the brackets be −, the sign of affection of every term placed within the brackets must be *changed.*

Thus we may express $a + b - c - d$ in any of the following ways:—

$$a + b - c - d = a + (b - c - d) = (a + b) - (c + d)$$
$$= a - (-b + c + d) = (a + b - d) - c, \text{ \&c.}$$

(When the word *sign* is used in future, the student is to understand *sign of affection,* unless otherwise expressed.)

Ex. V.

Simplify the expressions—

1. $3x - (2x - y) + (5x + 3y - 2z) - (7x - 4y - 3z)$.

2. $(2a^3 + 2a^2b + 2ab^2) - (2a^3 + a^2b + ab^2 - b^3) + (a^3 - a^2b - ab^2 - 2b^3)$.

3. $1 - (1 - 2x) - \{3 - (4 - 3x)\} + \{5 - (4x - \overline{7 - x})\}$.

4. $6x^2 - (2 - 3x + x^2) + \{-7 + (5x - \overline{8x - 2})\} - (3 - \overline{3 - 3x})$.

Group together the terms of the four following expressions, so that—

(i.) The first two and last two are inclosed in brackets.

(ii.) The last three are inclosed in brackets.

(iii.) The first three are so inclosed, and an inner pair of brackets used to inclose the second and third.

5. $a - b + c - d$.

6. $-6a + 7b - 3c + 5d$.

7. $-4x^3 + 12x^2y - 12xy^2 + 4y^3$.

8. $a^3 - b^3 - c^3 + 3abc$.

Add together—

9. $ax^2 + bxy + cy^2$, $-dx^2 - axy + ey^2$, $bx^2 + 3cxy + fy^2$, $-2ax^2 - 2by^2$.

10. $ax - cy - ez$, $-bx + dy + fz$, $cx - ey - gz$, $-dx + fy + hz$.

Subtract—

11. $(2a + b)x - (3b + c)y + (4c + d)z$ from $(3a - d)x + (4b - a)y + (5c - b)z$.

12. $(y - z)a^2 + (z - x)ab + (x - y)b^2$ from $(y - x)a^2 - (y - z)ab - (z - x)b^2$.

Show that the sum of:

13. $ax + by + cz$, $ay + bz + cx$, $az + bx + cy$ is $(a + b + c)(x + y + z)$.

14. $(a + b)x + (b + c)y + (a + c)z$, $(b + c)x + (a + c)y + (a + b)z$, and $(a + c)x + (a + b)y + (b + c)z$ is $2(a + b + c)(x + y + z)$.

15. $(a - b)x + (b - c)y + (c - a)z$, $(b - c)x + (c - a)y + (a - b)z$, and $(c - a)x + (a - b)y + (b - c)z$ is zero.

16. $3(a - b) - 2(c - d)$, $4(a - c) - 3(b - d)$, $7(d - a) - 6(c - b)$ is $12(d - c)$.

17. $2(a + b) - 3(c + d)$, $3(a + c) - 4(b + d)$, $(b + c) - 3(a + d)$ is $2a - 3b - 10d$.

18. $(x - y - z)x$, $(x - y - z)y$, $(y - z)z$, $(z + x)z$ is $x^2 - y^2$.

Prove that:

19. $3x - 5\left[7x - \overline{6x + 3\{9 - 4(2x + 2)}\}\right] = 15 - 122x.$

20. $(5 - 3x)\left\{9x - (5x - \overline{6 - 4x})\right\} = 30 - 18x.$

21. $(x - y)(z + u) + (y - z)(x + u) + (z - x)(y + u) = 0.$

22. $(a - b)(a + b) + (b - c)(b + c) + (c - a)(c + a) = 0.$

23. $7\left\{x - \overline{2x + 5}\right\} + 9\left\{\overline{3 - x} - 6(2 - 4x)\right\} = 200x - 116.$

24. $\dfrac{2x + 7}{3} + \dfrac{9 - 8x}{6} + \dfrac{5 + 6x}{9} = 4\dfrac{7}{18}.$

25. $4x - 3\left[7x - \left\{4x + 5(6x - \overline{3 + 5x})\right\}\right] = 10x - 45.$

26. $8\left\{x - (4x + 3)\right\} + \left\{6x - \overline{3(2x + 3) - 1}\right\}(3 - 2x) = -8x - 48.$

27. $\left\{5(x + y) - 3(x - y)\right\}\left\{2x + 4y + 2\,.\,\overline{4 - x - 2y}\right\} = 16x + 64y.$

28. $a - \left[2a - \left\{3a - (4a - \overline{5a + 7})\right\}\right] = 3a + 7.$

Multiplication.

24. Remembering the definition (Art. 14) of a coefficient, it follows that the product of two terms having coefficients is found by multiplying the product of the coefficients by the product of the remaining factors.

Thus, the product of $4a$ and $3b$, or $4a \times 3b = 12ab$.

Again, the terms to be multiplied may contain like letters.

Now, $a^3 = aaa$, $a^5 = aaaaa$, and hence it follows that

$$a^3 \times a^5 = aaa \times aaaaa = aaaaaaaa = a^8.$$

And, generally, we have

$$\begin{aligned} a^m \times a^n &= aaa\ldots\text{to } m \text{ factors} \times aaa\ldots\text{to } n \text{ factors}, \\ &= aaa\ldots\text{to } (m+n) \text{ factors} = a^{m+n}. \end{aligned}$$

It therefore follows that the product of the *powers* of like letters is found by adding together the *exponents* of the like letters.

Thus, $a^4 \times a^2 = a^6$, and $a^3 \times a = a^4$, &c.

And, further, as regards the sign of the product, we have seen, in Art. 12, that *like signs* give +, and *unlike signs* give −.

There are three things, then, which must be attended to in the multiplication of algebraical terms, viz.:—

1. The *signs*.—Like signs give +, and unlike signs give −.

2. The *coefficients*.—These are to be multiplied like ordinary numbers.

3. The *letters*.—The exponents of like letters are to be added together, and the powers so obtained written side by side with the unlike letters.

$$\begin{array}{rrrrrr} & \text{Ex. 1.} & & \text{Ex. 2.} & & \text{Ex. 3.} \\ & 6\,a^2b & & -\,3\,xyz & & -\,5\,abd \\ & \underline{2\,a^3c} & & \underline{x^2yz^3} & & \underline{-\,2\,bcd^2} \\ \text{Ans.} & 12\,a^5bc & \text{Ans.} & -\,3\,x^3y^2z^4 & \text{Ans.} & 10\,ab^2cd^3 \end{array}$$

25. Whenever the multiplicand or multiplier, or both, contains more than one term, it is evident the product is found by multiplying each term of the multiplicand separately by each term of the multiplier, and adding together the separate products.

$$\begin{array}{rll} & \text{Ex. 4.} & \text{Ex. 5.} \\ & 3\,a^2 - 7\,ab + 5\,b^2 & x^3 - 3\,x^2y + 3\,xy^2 - y^3 \\ & \underline{6\,ab\qquad\qquad\qquad} & \underline{-\,2\,xy\qquad\qquad\qquad\qquad} \\ \text{Ans.} & 18\,a^3b - 42\,a^2b^2 + 30\,ab^3 & -\,2\,x^4y + 6\,x^3y^2 - 6\,x^2y^3 + 2\,xy^4 \end{array}$$

Ex. 6.

$$\begin{array}{lllll}
x^2 & + xy & + y^2 & & \\
x^2 & - xy & + y^2 & & \\
\hline
x^4 & + x^3y & + x^2y^2 & & \\
 & - x^3y & - x^2y^2 & - xy^3 & \\
 & & x^2y^2 & + xy^3 & + y^4 \\
\hline
\text{Ans. } x^4 & & + x^2y^2 & & + y^4.
\end{array}$$

Ex. 7.

$$\begin{array}{lllll}
x^3 & + (a + b)\,x & + ab & & \\
x & + c & & & \\
\hline
x^3 & + (a + b)\,x^2 & + abx & & \\
 & cx^2 & + (ac + bc)\,x & + abc & \\
\hline
x^3 & + (a + b + c)\,x^2 & + (ab + ac + bc)\,x & + abc. &
\end{array}$$

It will be observed that the terms in the above examples are all arranged according to the power of some letter. Thus, in Ex. 4, they are arranged according to the descending powers of a; and in Exs. 5, 6, 7, they are arranged according to the descending powers of x. It matters not whether they are arranged according to ascending or descending powers, and the result would be the same if they are not so arranged. It is then, however, much easier to collect like terms, as they generally fall under each other. When we come to division we shall find it *necessary* to arrange the terms according to the power of some letter.

Ex. VI.

1. Multiply $3a + 2b$ by $4a - 3b$, and $6x + 7y$ by $3x - 5y$.

2. Multiply $x^2 + 2xy - 2y^2$ by $x - 2y$, and $15x^2 + 17xy - 4y^2$ by $2x + y$.

3. Multiply $a^2 + 2ab + b^2$ by $a^2 - 2ab + b^2$, and $a^2 + b^2$ by $a^2 - b^2$.

4. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$, and the product by $x^4 + y^4$.

5. Multiply $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.

6. Find the continued product of $x^4 + y^4$, $x^2 + y^2$, $x + y$, $x - y$.

7. Develop the expressions $(a + b)^3$, and $(x - y)^4$.

8. Multiply $5a^3 + 15a^2 + 45a + 135$ by $a^2 - 2a - 3$.

9. Multiply $1 + 3x - 7x^2$ by $x - 2$, and $a - x$ by $x^2 + ax + a^2$.

10. Multiply $a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4$ by $a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4$.

11. Multiply $a + bx + cx^2 + dx^3$ by $cx + f$.

12. Find the product of $x^2 + px + q$ and $x - a$.

13. Find the continued product of $x + a$, $x + b$, $x + c$.

Show that :

14. $\left(\frac{1}{2} - \frac{a}{3} + \frac{b}{4}\right)\left(\frac{1}{2} + \frac{a}{3} + \frac{b}{4}\right) = \frac{1}{4} - \frac{a^2}{9} + \frac{b}{4} + \frac{b^2}{16}.$

15. $\left(4x^2 - \frac{2}{5}x + \frac{1}{25}\right)\left(2x + \frac{1}{5}\right) = 8x^3 + \frac{1}{125}.$

16. $(x^4 + ax^3 + a^2x^2 + a^3x + a^4)(x^5 + a^5)(x - a) = x^{10} - a^{10}.$

17. $(x^2 + ax + a^2)(x^2 - ax + a^2)(x + a)(x - a) = x^6 - a^6.$

18. $(x + 1)(x + 3)(x + 4)(x - 8) = x^4 - 45x^2 - 140x - 96.$

19. $(3a^m + 4b^n)(4a^m - 3b^n) = 12a^{2m} + 7a^mb^n - 12b^{2n}.$

20. $(x^3 + 2ax^2 + 2a^2x + a^3)(x^3 - 2ax^2 + 2a^2x - a^3) = x^6 - a^6.$

21. $(b - c)(a^2 + ma + n) + (c - a)(b^2 + mb + n) + (a - b)(c^2 + mc + n) = ab(a - b) + bc(b - c) + ac(c - a).$

22. $(a^mx^n + b^px^q)(a^m - b^px^{q-n}) = a^{2m}x^n - b^{2p}x^{2q-n}.$

23. $(a + b + c + d)(a + b - c - d)(a - b - c + d)(a - b + c - d) + (b + c + d - a)(a + c + d - b)(a + b + d - c)(a + b + c - d) = 16abcd.$

24. $(a + b + c)(a + b - c) + (b + c - a)(c + a - b) = 4ab.$

25. $\left\{a^{(x-1)y} - b^{(y-1)x}\right\}(a^y + b^x) = a^{xy} - b^{xy} + a^{(x-1)y}b^x - a^yb^{(y-1)x}.$

26. $\left(\frac{1}{2}a^{\frac{1}{2}} + \frac{1}{3}b^{\frac{1}{4}} - \frac{1}{4}c^{\frac{1}{6}}\right)\left(\frac{1}{2}a^{\frac{1}{2}} - \frac{1}{3}b^{\frac{1}{4}} + \frac{1}{4}c^{\frac{1}{6}}\right) = \frac{1}{4}a - \frac{1}{9}b^{\frac{1}{2}} - \frac{1}{16}c^{\frac{1}{3}} + \frac{1}{6}b^{\frac{1}{4}}c^{\frac{1}{6}}.$

27. $\left(3a + 6b^{\frac{1}{2}} - 9c^{\frac{1}{3}} + 12d^{\frac{1}{4}}\right)\left(2a + 4b^{\frac{1}{2}} + 6c^{\frac{1}{3}} - 8d^{\frac{1}{4}}\right)$
$= 6\left(a^2 + 4b - 9c^{\frac{2}{3}} - 16d^{\frac{1}{2}} + 4ab^{\frac{1}{2}} + 24c^{\frac{1}{3}}d^{\frac{1}{4}}\right)$.

28. $(x^3 - y^3)(x + y) - (x^3 + y^3)(x - y) = 2x^3y - 2xy^3$.

29. $(2x - 3y)^3 + (4y - 5x)^3 + (3x - y)^3 - 3(2x - 3y)(4y - 5x)(3x - y) = 0$.

30. $(x + y)^4 + (x - y)^4 - 2(x^2 - y^2)^2 = 16x^2y^2$.

23. We have explained in the last Article the general method to be pursued in Multiplication. There are, however, many algebraical expressions which may be multiplied by inspection.

I. Expressions of the form $(a + b)^2$.

It is easily found by long multiplication that

$$(a + b)^2 = a^2 + 2ab + b^2,$$

and this, expressed in ordinary language, may be read thus :—

The square of the sum of two quantities is the square of the first, plus twice their product, plus the square of the second.

(This rule is evidently true whether the quantities are *positive* or *negative.*)

Ex. 1. $(3a + 7b)^2 = (3a)^2 + 2(3a)(7b) + (7b)^2 = 9a^2 + 42ab + 49b^2$.

Ex. 2. $(6x - 5)^2 = (6x)^2 + 2(6x)(-5) + (-5)^2 = 36x^2 - 60x + 25$.

Ex. 3. $(a + b + c)^2 = \left\{(a + b) + c\right\}^2$
$= (a + b)^2 + 2(a + b)c + c^2$
$= (a^2 + 2ab + b^2) + 2(ac + bc) + c^2$
$= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$

Remark.—It is a very common mistake with beginners to write down $a^2 + b^2$ as the square of $(a + b)$, thereby leaving out *twice the product of the quantities a and b.* They should impress this formula, viz. :—

$$(a + b)^2 = a^2 + 2ab + b^2$$

thoroughly upon the mind.

II. The form $(a + b)(a - b)$.

It may be easily found by long multiplication that

$$(a + b)(a - b) = a^2 - b^2,$$

which, in ordinary language, may be thus expressed :—

The product of the sum and difference of two quantities is the difference of the squares of the quantities.

This formula may be applied in all cases where the terms of the quantities to be multiplied are of the same magnitude, but when *some* of them differ in sign.

Ex. 1. $(a + b + c + d)(a + b - c - d)$

$$= \{(a+b)+(c+d)\}\{(a + b) - (c + d)\}$$
$$= (a + b)^2 - (c + d)^2 = (a^2 + 2ab + b^2) - (c^2 + 2cd + d^2)$$
$$= a^2 + b^2 - c^2 - d^2 + 2ab - 2cd.$$

Ex. 2. $(a^3 - 3a^2b + 3ab^2 - b^3)(a^3 + 3a^2b + 3ab^2 + b^3)$

$$= \{(a^3 + 3ab^2) - (3a^2b + b^3)\}\{(a^3 + 3ab^2) + (3a^2b + b^3)\}$$
$$= (a^3 + 3ab^2)^2 - (3a^2b + b^3)^2$$
$$= (a^6 + 6a^4b^2 + 9a^2b^4) - (9a^4b^2 + 6a^2b^4 + b^6)$$
$$= a^6 - 3a^4b^2 + 3a^2b^4 - b^6.$$

The principle to be adopted in all such cases is to find what terms in the given quantities are *exactly alike*, and put them *first* in brackets, when the remaining terms will fall into the proper form. Thus—

$$(3a + 7b + 5c - d)(3a - 7b + 5c + d) = \{(3a + 5c) + (7b - d)\}\{(3a + 5c) - (7b - d)\}.$$

And $(3a - 7b - 5c + d)(3a + 7b + 5c + d) = \{(3a + d) - (7b + 5c)\}\{(3a + d) + (7b + 5c)\}$.

(III.) The form $(x + a)(x + b)$.

By multiplication we find that—

$$(x + a)(x + b) = x^2 + (a + b)x + ab,$$

which in ordinary language may be thus expressed :—

The product of two binomials containing a common term and an uncommon term, is the square of the common term, plus the sum of the uncommon terms multiplied into the common term, plus the product of the uncommon term.

Thus—

$(x + 2)(x + 3) = x^2 + (2 + 3)x + 2 \times 3 = x^2 + 5x + 6.$

$(x + 6)(x - 1) = x^2 + (6 - 1)x + 6(-1) = x^2 + 5x - 6.$

$(x - 5)(x - 7) = x^2 - (5 + 7)x + (-5)(-7) = x^2 - 12x + 35.$

$(x + 3a)(x - 5a) = x^2 + (3a - 5a)x + (3a)(-5a) = x^2 - 2ax - 15a^2.$

And so,

$$(x + a + b)(x + c + d) = (x + \overline{a + b})(x + \overline{c + d})$$
$$= x^2 + (a + b + c + d)x + (a + b)(c + d).$$

We may extend this formula to any number of factors.

Thus, by multiplication, we find that—

$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc.$

$(x + a)(x + b)(x + c)(x + d) = x^4 + (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2 + (abc + abd + acd + bcd)x + abcd.$

Law of Formation of the Terms.

1. It will be seen that the coefficient of the first term is in each case *unity;* that of the second term, the sum of the uncommon letters taken *singly;* that of the third term, the sum of the uncommon letters taken *two together;* that of the fourth term, the sum of the uncommon letters taken *three together*, &c.

2. The power of the common letter is in the first term that of the number of the binomials, and it sinks *one* every term.

Ex. $(x + 1)(x + 2)(x + 3) = x^3 + (1 + 2 + 3)x^2 + (1 \times 2 + 1 \times 3 + 2 \times 3)x + 1 \times 2 \times 3 = x^3 + 6x^2 + 11x + 6.$

(IV.) The form $(a + b + c + d + \&c.)^2$.

By ordinary multiplication or otherwise we have $(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$; and a similar result follows if we take a larger number of terms.

We may hence deduce the following rule:—

The square of the sum of any number of quantities is the sum of the squares of the quantities together with twice the sum of the products formed by multiplying the first quantity into all that follow separately, then the second into all that follow, the third into all that follow, &c.

Ex. 1.
$$\begin{array}{rrrrrr} (1 + 2x + 3x^2)^2 = 1 & & + \; 4x^2 & & + \; 9x^4 \\ & 4x \; + & 6x^2 & + \; 12x^3 & \\ \hline = 1 & + \; 4x \; + & 10x^2 & + \; 12x^3 & + \; 9x^4. \end{array}$$

Ex. 2. $(a^3 + 3a^2b + 3ab^2 + b^3)^2$
$$\begin{array}{lrrrrrr} = a^6 & & + \; 9a^4b^2 & & + \; 9a^2b^4 & & \\ & 6a^5b \; + & 6a^4b^2 \; + & 2a^3b^3 & & & + \; b^6 \\ & & & 18a^3b^3 \; + & 6a^2b^4 & + \; 6ab^5 & \\ \hline = a^6 & + \; 6a^5b & + \; 15a^4b^2 & + \; 20a^3b^3 & + \; 15a^2b^4 & + \; 6ab^5 & + \; b^6. \end{array}$$

Ex. VII.

1. Find by inspection the squares of $x - y$, $3a - 5b$, $4c^2 + d^2$, $3x^2 - 2y^2$.

2. Find the continued product of $a + b$, $a^2 + b^2$, $a^4 + b^4$, and $a - b$.

3. Multiply $mx + ny$ by $mx - ny$, and $5a^2 - 3b^2$ by $10a^2 + 6b^2$.

4. Find the value of $(a + b + c + d)(a - b + c - d)$, and of $(a + b - c - d)(a + b + c + d)$.

5. Show that $(x^2 + 2xy + y^2)(x^2 - 2xy + y^2) = x^4 - 2x^2y^2 + y^4$.

6. Multiply $x + 5$ separately by $x - 1$, $x + 2$, $x - 3$, $x - 5$.

7. What is the continued product of $x - 2$, $x + 3$, $x - 6$, $x + 5$?

8. If $2s = a + b + c$, find the value of $s(s - a)(s - b)(s - c)$.

9. If $2s = a + b + c + d$, what is the value of—
$$(2s - 2a)^2 + (2s - 2b)^2 + (2s - 2c)^2 + (2s - 2d)^2?$$

Prove that—

10. $(ax + by)^2 + (cx + dy)^2 + (ay - bx)^2 + (cy - dx)^2 = (a^2 + b^2 + c^2 + d^2)(x^2 + y^2)$.

11. $\left\{(ac - bd)x + (ad + bc)y\right\}^2 + \left\{(ac - bd)y - (ad + bc)x\right\}^2 = (a^2 + b^2)(c^2 + d^2)(x^2 + y^2)$.

12. $(ax + by + cz)^2 = (a + b + c)(ax^2 + by^2 + cz^2) - ab(x - y)^2 - ac(x - z)^2 - bc(y - z)^2$.

13. $(x + a)(x + b)(x + c) = (x - a)(x - b)(x - c) + 2\left\{(a + b + c)x^2 + abc\right\}$.

14. $(m + n + p + q)^2 = (m + n)^2 + (m + p)^2 + (m + q)^2 + (n + p)^2 + (n + q)^2 + (p + q)^2 - 2(m^2 + n^2 + p^2 + q^2)$.

15. $(a^2 + b^2 + c^2)(m^2 + n^2 + p^2 + q^2) = (am + bn + cp)^2 + (an - bm + cq)^2 + (ap - bq - cm)^2 + (aq + bp - cn)^2$.

16. $(a - b)(b - c) + (b - c)(c - a) + (c - a)(a - b) = 3(ab + ac + bc) - (a + b + c)^2$.

17. $(x + y + z + a)^2 - (x - y - z + a)^2 = 4(x + a)(y + z)$.

18. $\left\{(a - b)^2 + 4ab\right\}\left\{(a + b)^2 - 4ab\right\}\left\{a^4 - b^4 + 2ab(a^2 - b^2)\right\} = (a + b)^5(a - b)^3$.

19. $(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2$.

20. $4(a^2 + b^2)(c^2 + d^2) = (a + b)^2(c + d)^2 + (a - b)^2(c - d)^2 + (a + b)^2(c - d)^2 + (a - b)^2(c + d)^2$.

21. $8(a^4 + b^4) = (a + b)^4 + 6(a^2 - b^2)^2 + (a - b)^4$.

22. $(x - y)^2 + (y - z)^2 + (z - x)^2 + 2(x - y)(y - z) + 2(x - y)(z - x) + 2(y - z)(z - x) = 0$.

23. $(a + b + c)(b + c - a)(a + c - b)(a + b - c) = 4a^2b^2$, when $a^2 + b^2 = c^2$.

24. $2\left\{(ax^m + by^n)^2 + (ay^n - bx^m)^2\right\} = (a^2 + b^2)\left\{(x^m + y^n)^2 + (x^m - y^n)^2\right\}$.

Division.

27. As in multiplication, we have especially three things to attend to, viz. :—

1. The *signs*.

We have learnt (Art. 13) that like signs give +, and unlike signs give −.

2. The *coefficients*.

Understanding here the *numerical* coefficients, it is plain that they may be divided as ordinary arithmetical quantities.

3. The *letters*.

As the product of the quantities a and b is expressed by ab, it follows that the quotient of ab by either of the factors a or b will give the remaining factor b or a respectively.

Thus, $ab \div a$ or $\frac{ab}{a} = b$, and $ab \div b$ or $\frac{ab}{b} = a$.

And so, $xyz \div xz = y$, and $pqrs \div qs = pr$.

We may then conclude that *when the divisor is contained as a factor in the dividend, the quotient is found by omitting from the dividend those of its factors which constitute the divisor.*

If the divisor be *not* contained as an exact factor in the dividend, we may then express the quotient symbolically.

Thus, $xy \div ab = \frac{xy}{ab}$.

When, however, the dividend and divisor have a common factor, it is plain that we may, as in arithmetic, strike out of the numerator and denominator of the symbolical quotient this common factor.

Thus, $5\,abc \div bd = \frac{5\,abc}{bd} = \frac{5\,ac}{d}$.

And $16\,xyz \div 10\,axz = \frac{16\,xyz}{10\,axz} = \frac{8\,y}{5\,a}$.

28. A power of a quantity is divided by any other power of the same quantity by subtracting the index of the divisor from that of the dividend, the quotient being that power of the quantity whose index is the remainder so obtained.

1. Let the power of the quantity in the dividend be the *higher*.

We have $a^5 = aaaaa$, and $a^3 = aaa$.

$$\therefore a^5 \div a^3 = \frac{aaaaa}{aaa} = aa = a^2 = a^{5-3}.$$

Or, generally, m being greater than n, since

$a^m = aaa\ldots$to m factors, and $a^n = aaa\ldots$to n factors,

we have—

$$a^m \div a^n = \frac{aaa\ldots\text{to } m \text{ factors}}{aaa\ldots\text{to } n \text{ factors}} = aaa\ldots(m - n) \text{ factors} = a^{m-n}.$$

2. Let the power of the quantity in the dividend be the *lower*.

Suppose we have to divide a^4 by a^7.

We have $a^4 \div a^7 = \dfrac{aaaa}{aaaaaaa} = \dfrac{1}{aaa} = \dfrac{1}{a^3}$.

We may, however, so express the result that it shall agree exactly with the proposition at the head of this article.

For we may conceive of a^3 as representing the *product* of *unity* and the quantity a^3. We shall therefore be perfectly consistent if we allow a^3 to represent the *quotient* of *unity* by the quantity a^3.

We shall then have $a^{-3} = \dfrac{1}{a^3}$, and hence we get from the above result—

$$a^4 \div a^7 = a^{-3} = a^{4-7}.$$

Or, generally, m being less than n, we have—

$$\frac{a^m}{a^n} = \frac{aaa\ldots\ldots\text{to } m \text{ factors}}{aaa\ldots\ldots\text{to } n \text{ factors}} = \frac{1}{aaa\ldots\ldots(n - m)\text{ factors}}$$

$$= \frac{1}{a^{n-m}}\text{; or, using the notation just explained,}$$

$$= a^{-(n-m)} = a^{-n+m} = a^{m-n}.$$

3. Let the powers of the quantities in the dividend and divisor be *equal.*

It is evident that their quotient is *unity.*

Thus, $\dfrac{a^4}{a^4} = \dfrac{aaaa}{aaaa} = \dfrac{1}{1} = 1.$

And so, $\dfrac{a^m}{a^m} = 1.$

If, however, we *assume* the principle proved in the two cases above to hold here, we have—

$$\frac{a^m}{a^m} = a^{m-m} = a^0.$$

It follows therefore that $a^0 = 1$.

COR.—From the above interpretation of negative indices it follows that the same rules for multiplication and division of quantities involving them may be applied as in the case of positive indices.

Thus, $a^5 \times a^{-2} = a^5 \times \dfrac{1}{a^2} = \dfrac{a^5}{a^2} = a^{5-2}$.

And so, $a^4 \div a^{-3} = a^{4-(-3)} = a^7$.

Ex. 1. $\dfrac{20\,x^2yz}{4\,xz} = 5\,xy.$

Ex. 2. $\dfrac{-\,45\,a^3b^2c^4}{15\,ab^2c^2} = -\,3\,a^2c^2.$

Ex. 3. $\dfrac{-\,7\,a^3 + 21\,a^2b - 14\,ab^2}{-\,7\,a} = a^2 - 3\,ab + 2\,b^2.$

Ex. 4. $\dfrac{4\,x^5yz^3 - 6\,x^3y^3z^4 + 42\,xy^4z^4}{3\,x^2y^4z^5}$

$$= \frac{4\,x^3}{3\,y^3z^2} - \frac{2}{yz} + \frac{14}{xz},\text{ or}$$

$$= \frac{4}{3}x^3y^{-3}z^{-2} - 2\,y^{-1}z^{-1} + 14\,x^{-1}z^{-1}$$

Ex. 5. Divide $3\,a^2 + 13\,ab - 10\,b^2$ by $a + 5\,b$.

When the divisor, as in this example, contains more than one term, it is generally convenient to follow the method of arithmetical long division. Thus—

$$\begin{array}{rl}
a + 5\,b)\; & 3\,a^2 + 13\,ab - 10\,b^2\,(3\,a - 2\,b \\
 & \underline{3\,a^2 + 15\,ab\phantom{{}-10\,b^2}} \\
 & -\,2\,ab - 10\,b^2 \\
 & \underline{-\,2\,ab - 10\,b^2}
\end{array}$$

Ex. 6. Divide $a^4 + a^2b^2 + b^4$ by $a^2 + ab + b^2$.

$$\begin{array}{rl}
a^2 + ab + b^2)\; & a^4 + a^2b^2 + b^4\,(a^2 - ab + b^2 \\
 & \underline{a^4 + a^3b + a^2b^2} \\
 & -\,a^3b + b^4 \\
 & \underline{-\,a^3b - a^2b^2 - ab^3} \\
 & a^2b^2 + ab^3 + b^4 \\
 & \underline{a^2b^2 + ab^3 + b^4}
\end{array}$$

It will be seen that in the last two examples care has been taken to keep the terms of the divisor, dividend, and successive remainders arranged according to the ascending or descending powers of some letter. In these cases we have arranged the terms according to the descending powers of a, and, as therefore follows, according to the ascending powers of b. Want of care in this respect will often render the operation of finding the true quotient *tiresome,* if not impossible. The next two examples will illustrate this point. They may be attempted first by the student, keeping the terms in the order as given.

Ex. 7. Divide $2 - 7x - 15x^2$ by $5x - 1$.

$$\begin{array}{rl}
-1+5x) & 2-7x-15x^2\,(-2-3x \\
 & \underline{2-10x} \\
 & \phantom{2-{}}3x-15x^2 \\
 & \phantom{2-{}}\underline{3x-15x^2}
\end{array}$$

Or thus,

$$\begin{array}{rl}
5x-1) & -15x^2-7x+2\,(-3x-2 \\
 & \underline{-15x^2+3x} \\
 & -10x+2 \\
 & \underline{-10x+2}
\end{array}$$

Ex. 8. Divide $x^3 + y^3$ by $x^{-1} + y^{-1}$.

Here we have the powers of x in the dividend descending, while in the divisor they are ascending. Arranging them in the divisor as in the dividend, the operation is easy. Thus—

$$\begin{array}{rl}
y^{-1}+x^{-1}) & x^3+y^3\,(x^3y-x^2y^2+xy^3 \\
 & \underline{x^3+x^2y} \\
 & -x^2y+y^3 \\
 & \underline{-x^2y-xy^2} \\
 & \phantom{x^3-x^2y-{}}xy^2+y^3 \\
 & \phantom{x^3-x^2y-{}}\underline{xy^2+y^3}
\end{array}$$

We shall work the next example in two ways to illustrate, firstly, the above point again; and, secondly, to show how the operation may be sometimes abbreviated by the use of brackets.

Ex. 9. Divide $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.

$$\begin{array}{rl}
x+y+z) & x^3-3xyz+y^3+z^3\,(x^2-xy-xz+y^2-yz+z^2 \\
 & \underline{x^3+x^2y+x^2z} \\
 & -x^2y-x^2z-3xyz \\
 & \underline{-x^2y-xy^2-xyz} \\
 & -x^2z+xy^2-2xyz \\
 & \underline{-x^2z-xyz-xz^2} \\
 & \phantom{x^3-x^2y-x^2z+{}}xy^2-xyz+xz^2+y^3 \\
 & \phantom{x^3-x^2y-x^2z+{}}\underline{xy^2+y^3+y^2z} \\
 & -xyz+xz^2-y^2z \\
 & \underline{-xyz-y^2z-yz^2} \\
 & \phantom{x^3-x^2y-x^2z+xy^2-xyz+{}}xz^2+yz^2+z^3 \\
 & \phantom{x^3-x^2y-x^2z+xy^2-xyz+{}}\underline{xz^2+yz^2+z^3}
\end{array}$$

Or thus, inclosing the last two terms of the divisor in brackets—

$$
\begin{array}{rl}
x+(y+z)\,\big) & x^3-3xyz+y^3+z^3\,\big(\,x^2-(y+z)x+(y^2-yz+z^2) \\
 & \underline{x^3+(y+z)\,x^2} \\
 & \quad -(y+z)\,x^2-3\,xyz \\
 & \quad \underline{-(y+z)\,x^2-(y^2+2\,yz+z^2)\,x} \\
 & \qquad\qquad (y^2-yz+z^2)\,x+y^3+z^3 \\
 & \qquad\qquad \underline{(y^2-yz+z^2)\,x+y^3+z^3}
\end{array}
$$

It will be seen in both the above operations that we have brought down the terms of the dividend only when the subtrahends indicated they were required. This often prevents much useless repetition.

Ex. VIII.

Find the quotient of—

1. $28\,a^2b - 7\,ab^2 + 14\,b^3$ by $7\,b$; $3\,x^2y^2 - 12\,xy^3$ by $3\,xy$.
2. $-\,6\,a^5b + 15\,a^4b^2 - 20\,a^3b^3$ by $-\,3\,ab$; $4\,x^3y^2 + 6\,x^2y^3 + 4\,xy^4$ by $2\,xy^2$.
3. $ax^{2m} + bx^my^n + cy^{2n}$ by x^{m+n}; $ax^{2m-2n} + bx^{m-n}y^n + cy^{2n}$ by x^{m-n}.
4. $30\,x^2 + 2\,xy - 12\,y^2$ by $5\,x - 3\,y$; and by $6\,x + 4\,y$.
5. $1 + 2\,x + 3\,x^2 + 2\,x^3 + x^4$ by $1 + x + x^2$.
6. $12 - 19\,x - 21\,x^2$ by $7\,x - 3$; and by $3\,x + 4$.
7. $x^4 - 4\,x^2y^2 + 12\,xy^3 - 9\,y^4$ by $x - 3\,y$; and by $x + y$.
8. $x^5 - y^5$ by $x - y$; and $x^5 + y^5$ by $x + y$.
9. $acx^{m+n} + adx^my^m + bcx^ny^n + bdy^{m+n}$ by $cx^n + dy^m$.
10. $a^6 - a^4b^2 - a^2b^4 + b^6$ by $a^3 - a^2b - ab^2 + b^3$.
11. $b^2 + ab + bc + ac$ by $a + b$; and $a^2 + ab + bc + ac$ by $a + c$.
12. $a + (a + b)\,x + (a + b + c)\,x^2 + (a + b + c)\,x^3 + (b + c)\,x^4 + cx^5$ by $1 + x + x^2 + x^3$.
13. $a^5 - pa^4 + qa^3 - qa^2 + pa - 1$ by $a - 1$.

14. $a^4 + b^4 + c^4 + d^4 - 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) - 8\,abcd$ by $a - b + c + d$; and that of this quotient by $a - b - c - d$.

15. Show that the remainder, after the division of $x^4 - px^3 + qx^2 - rx + s$ by $x - a$, is $a^4 - pa^3 + qa^2 - ra + s$.

16. Divide $x^4 - y^4$ by $x^{-2} - y^{-2}$, and $x^2 + x^{-2} + 2$ by $x + x^{-1}$.

17. Show that the quotient of 1 by $1 + x$, is $1 - x + x^2 - x^3 +$ &c. *ad infinitum.*

18. Show that the quotient of 1 by $1 - 3\,x + 3\,x^2 - x^3$ is $1 + 3\,x + 6\,x^2 + 10\,x^3 + 15\,x^4 +$ &c. *ad infinitum.*

19. Divide $(x + y)^4 - 2\,(x + y)^2\,z^2 + z^4$ by $x + y - z$.

20. Divide $(a - c)^3 - 3\,(a - c)^2\,(b - d) + 3\,(a - c)\,(b - d)^2 - (b - d)^3$ by $a - b - c + d$.

Prove that:

21. $\{ab\,(b - a) + ac\,(a - c) + bc\,(c - b)\} \div (b - c) = (a - b)\,(c - a)$.

22. $\{a^2\,(b^2 - c^2) - ab\,(2\,b^2 + bc - c^2) + b^3\,(b + c)\} \div (ab - ac - b^2) = (a - b)\,(b + c)$.

23. $\dfrac{ab + 2\,a^2 - 3\,b^2 - 4\,bc - ac - c^2}{2\,a + 3\,b + c} = a - b - c.$

24. $\dfrac{12\,a^2 + 7\,ab - 12\,b^2 + ac - 7\,bc - c^2}{3\,a + 4\,b + c} = 4\,a - 3\,b - c.$

25. $\dfrac{x^3 - 2\,ax^2 + (a^2 - ab - b^2)\,x + a^2b + ab^2}{(x - a)\,(x + b)} = x - a - b.$

26. $\dfrac{(x^2 - 1)(y^2 - 1)(z^2 - 1) + (x + yz)(y + xz)(z + xy)}{x^2 + y^2 + z^2 + 2\,xyz - 1} = xyz + 1.$

27. $\dfrac{x^3\,(y - z) + y^3\,(z - x) + z^3\,(x - y)}{(x - y)\,(y - z)\,(x - z)} = x + y + z.$

28. $\dfrac{x^4\,(y - z) + y^4\,(z - x) + z^4\,(x - y)}{(x - y)\,(y - z)\,(x - z)} = x^2 + y^2 + z^2 + xy + yz + xz.$

29. $\dfrac{abcd\,(a+b+c+d)^2-(abc+bcd+cda+dab)^2}{(bc-ad)\,(ca-bd)}=cd-ab.$

30. $\dfrac{(a+b)^3+(a+c)^3+(b+c)^3-3\,(a+b)(a+c)(b+c)}{a^2+b^2+c^2-ab-ac-bc}=2(a+b+c).$

31. $\dfrac{(x-y)^3+(y-z)^3}{x-z}=(x-y)^2+(y-z)^2-(x-y)\,(y-z).$

32. $\dfrac{x^{mn}-1}{x^m-1}=x^{(n-1)m}+x^{(n-2)m}+\ldots\ldots+x^{2m}+x^m+1.$

33. $\dfrac{a^4-b^3}{a^{\frac{1}{3}}-b^{\frac{1}{4}}}=a^{\frac{11}{3}}+a^{\frac{10}{3}}b^{\frac{1}{4}}+a^3b^{\frac{1}{2}}+\ldots\ldots+a^{\frac{1}{3}}b^{\frac{11}{4}}+b^3.$

34. $(x-y)\div(\sqrt[5]{x}-\sqrt[5]{y})=x^{\frac{4}{5}}+x^{\frac{3}{5}}y^{\frac{1}{5}}+x^{\frac{2}{5}}y^{\frac{2}{5}}+x^{\frac{1}{5}}y^{\frac{3}{5}}+y^{\frac{4}{5}}.$

Factors.

29. The *ordinary* method of finding the quotient of two algebraical quantities having been explained in the last article, we shall now proceed to show how, in certain cases, this method may be avoided, and the quotient written down at sight. It may be remarked at the outset, that the resolution of algebraical expressions into their elementary factors is a subject of very great importance, and one which the student will do well to thoroughly master.

(I.) The form $x^2 + 2\,ax + a^2$.

We have seen (Art. 26) that $x^2 + 2\,ax + a^2 = (x + a)^2$. Hence the *sum of the squares* of two quantities, together with twice the product of the quantities, is equal to the *square of the sum* of the quantities.

And hence, any algebraical expression, which can be thrown into the form $(x^2 + 2\,ax + a^2)$, is of necessity a perfect square.

Thus—

$x^2 + 6\,x + 9 = x^2 + 2\,(3)\,x + 3^2 = (x + 3)^2.$

$a^2 - 10\,ab + 25\,b^2 = a^2 + 2\,a\,(-5\,b) + (-5\,b)^2 = (a - 5\,b)^2.$

$16\,a^2x^2 - 56\,abxy + 49\,b^2y^2 = (4\,ax)^2 + 2\,(4\,ax)\,(-7\,by) + (-7\,by)^2 = (4\,ax - 7\,by)^2.$

$3\,ab^2x^2 - 6\,abcxy + 3\,ac^2y^2 = 3\,a\,(b^2x^2 - 2\,bcxy + c^2y^2) = 3\,a\left\{(bx)^2 - 2\,(bx)\,(cy) + (cy)^2\right\} = 3\,a\,(bx - cy)^2.$

(II.) The form $a^2 - b^2$.

We have seen (Art. 26, II.) that $a^2 - b^2 = (a + b)\,(a - b)$.

Hence the *difference of the squares* of two quantities is equal to the product of the sum and difference of the quantities. Thus—

$$a^2x^2 - b^2y^2 = (ax)^2 - (by)^2 = (ax + by)\,(ax - by).$$

$$\begin{aligned} x^4 - y^4 &= (x^2)^2 - (y^2)^2 = (x^2 + y^2)\,(x^2 - y^2) \\ &= (x^2 + y^2)\,(x + y)\,(x - y). \end{aligned}$$

$$\begin{aligned} a^2 + b^2 - c^2 - d^2 + 2\,ab - 2\,cd& \\ &= (a^2 + 2\,ab + b^2) - (c^2 + 2\,cd + d^2) \\ &= (a + b)^2 - (c + d)^2 \\ &= \left\{(a + b) + (c + d)\right\}\left\{(a + b) - (c + d)\right\} \\ &= (a + b + c + d)\,(a + b - c - d). \end{aligned}$$

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= (x^4 + 2\,x^2y^2 + y^4) - x^2y^2 \\ &= (x^2 + y^2)^2 - x^2y^2 \\ &= \left\{(x^2 + y^2) + xy\right\}\left\{(x^2 + y^2) - xy\right\} \\ &= (x^2 + xy + y^2)\,(x^2 - xy + y^2). \end{aligned}$$

(III.) The form $x^2 + px + q$.

This form evidently includes both the preceding, for the first form—viz., $x^2 + 2\,ax + a^2$ is included, since q may be the square of half p; and the second is included—viz., $x^2 - a^2$, since we may have $p = o$, and q a negative square quantity.

Now, the resolution into elementary rational factors of the quantity $x^2 + px + q$ is not always possible; but, since (Art. 26, III.),

$$x^2 + (a + b)\,x + ab = (x + a)\,(x + b),$$

we have the following rule, when the quantity admits of resolution.

RULE.—If the third term q of the quantity $x^2 + px + q$ can be broken up into two factors, a and b, such that the sum of these will give the coefficient of x, then the elementary factors of $x^2 + px + q$ are $x + a$ and $x + b$.

Thus, $x^2 + 7\,x + 12 = (x + 3)\,(x + 4)$; for the *product* of 3 and 4 is 12, and their *sum* is 7, the coefficient of x.

$x^2 - x - 30 = (x - 6)(x + 5)$; for the *product* of -6 and 5 is -30, and their *sum* is -1.

And so, $x^2 - 18x + 32 = (x - 2)(x - 16)$,

And $a^2 + 3ab - 108b^2 = a^2 + (12b - 9b)a + (12b)(-9b) = (a + 12b)(a - 9b)$.

(IV.) The form $ax^2 + bx + c$.

This is the *general* form of a trinomial. The following remarks, though equally applying to each of the three preceding forms, are especially intended to be *practically* applied to trinomials not included by them.

The above form will include such expressions as the following:—$20x^2 + 11x - 42$, $6x^2 - 37x + 55$.

It is evident that the product of the first terms of the factors will be the first term of the given trinomial, and that the product of the last terms of the factors will be the third term of the given trinomial.

And, further, when the third term is *negative*, the last term of one factor must have the sign $+$, and the last term of the other the sign $-$; but, when the third term is *positive*, the last terms of the factors must have the same sign as the middle term.

Thus, $12x^2 - 31x - 30 = (4x + 3)(3x - 10)$.

Here the factors of $12x^2$ are either $3x$ and $4x$, $6x$ and $2x$, $12x$ and x, and the factors of 30 either 5 and 6, 3 and 10, 2 and 15, 1 and 30; and we must give a $+$ sign to one of each of these latter pairs, and a $-$ sign to the other. It is easily found on trial that, in order to obtain $-31x$ as the middle term, the factors of the trinomial must be $4x + 3$ and $3x - 10$.

So we have $10a^2 - 41ab + 21b^2 = (5a - 3b)(2a - 7b)$, and $acx^2 + (ad + bc)xy + bdy^2 = (ax + by)(cx + dy)$.

(V.) The forms $x^n + y^n$ and $x^n - y^n$.

We shall show in the next article that a rational integral algebraical expression, involving x, contains $x - a$ as a factor when it vanishes on substituting a for x.

Hence, $x^n + y^n$ and $x^n - y^n$ must each vanish on putting y for x, if they contain $x - y$ as a factor, n being an integer.

The former becomes $y^n + y^n$ or $2y^n$, and the latter $y^n - y^n$ or 0. We therefore conclude that—

$x^n + y^n$ does not contain $x - y$ as a factor, and that—

$x^n - y^n$ does contain $x - y$ as a factor, whether n be *even* or *odd*.

Again, on the same principle, they must each vanish if they contain $x + y$ as a factor, on putting $-y$ for x.

The former becomes $(-y)^n + y^n$, which vanishes when n is *odd*, and the latter becomes $(-y)^n - y^n$, which vanishes when n is *even*.

Hence we conclude that—

$x^n + y^n$ contains $x + y$ as a factor when n is *odd*, and
$x^n - y^n$ contains $x + y$ as a factor when n is *even*.

Now, the quotient of either of these quantities by $x + y$ or $x - y$ can in any particular case be found by long division. We thus find that—

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4),$$
$$x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3),$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

The law of formation of the co-factor in each case is easy to see; and if we may *assume* this apparent law as generally true, we may conclude that, when an algebraical quantity is of the form $x^n + y^n$ or $x^n - y^n$, and it contains $x + y$ or $x - y$ as a factor, the law of formation of the co-factor is as follows:—

Law of Formation of Co-Factor.

1. The terms are homogeneous, and of dimensions one degree lower than the given expression, the power of x in the first term being $n - 1$, and diminishing each successive term by *unity;* and the power of y increasing each successive term by *unity,* and first appearing in the second term.

2. The coefficient of every term is *unity.*

3. The *signs* are alternately + and −, when $x + y$ is the corresponding elementary factor; and are all +, when $x - y$ is the corresponding elementary factor.

Ex. 1. $a^5 + 32 = a^5 + 2^5 = (a + 2)(a^4 + a^3 \cdot 2 + a^2 \cdot 2^2 + a \cdot 2^3 + 2^4) = (a + 2)(a^4 + 2a^3 + 4a^2 + 8a + 16)$.

Ex. 2. $a^6 - b^6 = (a^3)^2 - (b^3)^2 = (a^3 + b^3)(a^3 - b^3) = (a + b)(a^2 - ab + b^2) \cdot (a - b)(a^2 + ab + b^2 = (a + b)(a - b)(a^2 - ab + b^2)(a^2 + ab + b^2)$.

30. *The remainder of the division of a rational integral function of* x *by* x − a *may be found by putting* a *for* x *in the given function.*

DEF.—A **function** of x is an algebraical expression involving x; and a rational integral function of x is an expression of the form $ax^n + bx^{n-1} + \&c. + sx + t$, where all the powers of x are integral and positive.

Let $f(x)$* be a rational integral function of x, and suppose Q to be the quotient, and R the remainder on dividing the function by $x - a$.

Then, evidently—

$$Q(x - a) + R = f(x) \text{ identically.}$$

And this identity must hold for all values of x, and therefore holds when $x = a$.

In this case we have $Q(a - a) + R = f(a)$

$$0 + R = f(a)$$

$$\text{or } R = f(a).$$

Now $f(a)$ is the result of putting a for x in the given function, and is, as we have just shown, the remainder on dividing the given function by $x - a$.

COR. 1. When there is no remainder, we must, of course, have $f(a) = 0$. Hence, a given rational integral function of x vanishes when a is put for x, if it be divisible by $x - a$.

Ex. 1. The remainder, after the division of $2x^3 - 5x^2 + 6x + 7$ by $x - 2$ is 15.

For, putting $x = 2$, we have—

$$2x^3 - 5x^2 + 6x + 7 = 2\cdot 2^3 - 5\cdot 2^2 + 6\cdot 2 + 7 = 15.$$

Ex. 2. The function—

$x^3 - 2x^2 + 5x - 52$ is divisible by $x - 4$.

For, putting $x = 4$, we have—

$$x^3 - 2x^2 + 5x - 52 = 4^3 - 2\cdot 4^2 + 5\cdot 4 - 52 = 0.$$

COR. 2. Any rational integral function of x is divisible by $x - 1$, when the sum of the coefficients of the terms is *zero*.

For, putting $x = 1$ in the given function, it is evident that it is reduced to the sum of its coefficients, which sum must be zero if the function be divisible by $x - 1$.

Ex. Each of the following functions is divisible by $x - 1$, viz.:—

$3x^4 + 7x^3 - x^2 + 12x - 21$, $5x^7 - 2x - 3$,

$(a-b)x^2 + (b-c)x + (c - a)$, $(a+b)^2x^2 - 4abx - (a-b)^2$.

---

* The expression $f(x)$ must not be considered to mean the product of f and x, but as a *symbol* used for convenience.

COR. 3. Any rational integral function of x is divisible by $x + 1$, when the sum of the coefficients of the *even* powers of x is equal to the sum of the coefficients of the *odd* powers.

(The term independent of x is always to be considered as the coefficient of an *even* power).

Let $ax^n + ax^{n-1} + \&c. + rx^2 + sx + t$ be a rational integral function of x.

Put $x = -1$, then we have, if the function be divisible by $x + 1$—

$$a(-1)^n + b(-1)^{n-1} + \&c. + r(-1)^2 + s(-1) + t = 0.$$

Suppose n to be even, then evidently $(-1)^n = (-1)(-1)(-1)\ldots$to an *even* number of factors $= +1$.

And so $(-1)^{n-1} = (-1)(-1)(-1)\ldots$to an *odd* number of factors $= -1$; and so on.

Hence we get $a - b + \&c. + r - s + t = 0$, and this must evidently require the condition that the sum of the positive quantities is equal to the sum of the negative, and, therefore, that the sum of the coefficients of the *even* powers of x is equal to the sum of the coefficients of the *odd* powers. And a similar result will follow if we suppose n to be *odd*.

Ex. Each of the following functions is divisible by $x + 1$, viz.:—

$x^3 + 5x^2 + 7x + 3$, $5x^5 - 4x^4 + 8x^2 - 2x - 1$,
$a^2x^4 - (a + 1)(a + 2)x^2 + 2x + 3a + 4$, $px^3 + (q + r)x^2 + (q + r)x + p$.

Ex. IX.

Resolve into elementary factors—

1. $x^2 - 9a^2$, $16y^4 - 25z^4$, $24a^2 - 54b^2$, $8x^3 - 27y^3$.
2. $x^4 - xy^3$, $a^8 - b^8$, $xy^4 + x^4y$, $2x^3y^2z - 8xy^2z^3$.
3. $a^4 - 4b^4$, $x^4 + x^2y^2 + y^4$, $a^4 - 2a^2b^2 + b^4$, $a^2 + b^2 - c^2 + 2ab$.
4. $a^2 + b^2 - c^2 - d^2 + 2ab - 2cd$, $a^2 - b^2 - c^2 + d^2 + 2bc + 2ad$, $a^2 - (b - c)^2$.
5. $(x + 7)^2 - (x + 2)^2$, $(x + 5)^2 - (x + 2)^2$, $(2a + b)^2 - (a - c)^2$.
6. $(x^3 - y^3)^2 + 4(x^4 + x^2y^2 + y^4)x^2y^2$, $(x^2 + y^2)^4 - 5(x^2 + y^2)^2x^2y^2 + 4x^4y^4$.

7. $x^2 - 3x - 70$, $x^2 + 11x + 10$, $a^2 - 15ab + 56b^2$, $x^2 - 4x - 192$.

8. $a^2x^2 + abxy - 42b^2y^2$, $3ax^2 - 24ax - 60a$, $24ac - 5ac^2 + ac^3$.

9. $6x^2 - 11x - 35$, $8x^2 + 6x - 135$, $18x^2 - 21x - 72$, $20x^2 - 11x - 42$.

10. $3x^3y + 10x^2y^2 + 3xy^3$, $20x^3 + 12ax^2 + 25bx^2 + 15abx$, $m^2x^2 + (mq + mp)x + pq$.

Write down the quotient of—

11. $x^4 - 16$ by $x - 2$, $3x^5 + 96$ by $x + 2$, $x^6 - 27$ by $x^2 - 3$.

12. $(a + b)^3 - (c + d)^3$ by $a + b + c + d$, $a^2 - b^2 + c^2 - d^2 + 2ac + 2bd$ by $a + b + c - d$.

Find the remainder after the division of—

13. $x^4 + px^3 + qx^2 + rx + s$ by $x - a$, $x^4 + a^4$ by $x^2 - a^2$.

14. $x^3 - 5x^2 + 7x - 9$ by $x + 3$, $x^4 - 3x + 7$ by $x - 2$.

Show that—

15. $5x^5 - 3x^3 + 7x^2 - 8x - 1$ is divisible by $x - 1$.

16. $2x^4 - 3x^3 + x^2 - 7x - 13$ is divisible by $x + 1$.

Show that:

17. $a^2(b - c) + b^2(c - a) + c^2(a - b) = (a - b)(b - c)(a - c)$.

18. $bc(b - c) + ac(c - a) + ab(a - b) = (a - b)(b - c)(a - c)$.

19. $a^3(b - c) + b^3(c - a) + c^3(a - b) = (a - b)(b - c)(a - c)(a + b + c)$.

20. $a^4(b - c) + b^4(c - a) + c^4(a - b) = (a - b)(b - c)(a - c)(a^2 + b^2 + c^2 + ab + ac + bc)$.

21. $a^3 + b^3 + c^3 + 3(b + c)(c + a)(a + b) = (a + b + c)^3$.

22. $x^6 - y^6 = (x^2 - xy + y^2)(x^2 - xy + y^2)(x + y)(x - y)$.

23. $x^8 - y^8 = (x^4 + y^4)(x^2 + y^2)(x + y)(x - y)$.

24. $x^2 - 2xy - 3y^2 + 4yz - z^2 = (x - 3y + z)(x + y - z)$.

25. $3(x+y)^2 + 2(x+y)(z+u) - 5(z+u)^2 = (x+y-z-u)(3x+3y+5z+5u)$.

26. $\frac{1}{5}\left\{(x-y)^5 + (y-z)^5 + (z-x)^5\right\}$
$= \frac{1}{2}\left\{(x-y)^2 + (y-z)^2 + (z-x)^2\right\} \cdot \frac{1}{3}\left\{(x-y)^3 + (y-z)^3 + (z-x)^3\right\}$.

27. $\left\{(x-y)^2 + (y-z)^2 + (z-x)^2\right\}^2$
$= 2\left\{(x-y)^4 + (y-z)^4 + (z-x)^4\right\}$.

28. $\left\{(x-y)^2 + (y-z)^2 + (z-x)^2\right\} . \frac{(x-y)^5 + (y-z)^5 + (z-x)^5}{5}$
$= \left\{(x-y)^4 + (y-z)^4 + (z-x)^4\right\} . \frac{(x-y)^3 + (y-z)^3 + (z-x)^3}{3}$.

29. $x^4(y^2 - z^2) + y^4(z^2 - x^2) + z^4(x^2 - y^2) = (x-y)(y-z)(z-x)(x+y)(y+z)(z+x)$.

30. $x^{2^n} - y^{2^n} = (x-y)(x+y)(x^2-y^2)(x^4+y^4)\ldots\ldots(x^{2^{n-1}} + y^{2^{n-1}})$.

31. $(a+b+c+d)^3 - (b+c-a-d)(c+a-b-d)(a+b-c-d) = 4(a+b+c+d)(ab+bc+ac+ad+bd+cd) - 8(abc+abd+acd+bcd)$.

CHAPTER III.

INVOLUTION AND EVOLUTION.

Involution.

31. Involution is the operation by which we obtain the *powers* of quantities. This can of course be done by multiplication, but the results obtained by the actual multiplication of simple forms enable us to develop without multiplication more complex forms. As the subject requires the aid of the Binomial Theorem, we shall here show how to develop a few only of the more simple expressions.

32. The power of a single term is obtained by raising the coefficient of the term to the power in question, and *multiplying* the exponents of the letters of the term by the exponent of the power in question.

Thus, $(a^3)^2 = a^3 \times a^3 = a^{3+3} = a^{3 \times 2}$,
$(a^m)^n = a^m \times a^m \times a^m \ldots\ldots \text{to } n \text{ factors} = a^{m+m+m+\ldots \text{to } n \text{ terms}} = a^{mn}$.
And so, $(4\,a^2b^3c^4)^3 = 4^3\,a^{2\times 3}b^{3\times 3}c^{4\times 3} = 64\,a^6b^9c^{12}$

33. Development of the *third*, *fourth*, and *fifth* powers of $a + b$.
We know (Art. 26, I.) that $(a + b)^2 = a^2 + 2\,ab + b^2$.
$\therefore (a + b)^3 = (a^2 + 2\,ab + b^2)(a + b) = a^3 + 3\,a^2b + 3\,ab^2 + b^3$;
also, $(a + b)^4 = (a^3 + 3\,ab^2 + ab^2 + b^3)(a + b) = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4$;
and $(a + b)^5 = (a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4)(a + b)$
$= a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5$.

The following law of the formation of the terms is evident:—

Law of Formation of Terms.

1. The first term contains a raised to the given power, and the power of a decreases by *unity* in each successive term, while the power of b (which first appears in the second term) increases by unity in each successive term, till it reaches the power of the given quantity.

2. The first coefficient is unity, and the coefficient of any term is found by multiplying the previous coefficient by the exponent of a in the previous term, and dividing the product by the number of terms hitherto developed.

Ex. 1. $(2\,x + 3\,y)^3 = (2\,x)^3 + 3\,(2\,x)^2\,(3\,y) + 3\,(2\,x)\,(3\,y)^2 + (3\,y)^3 = 8\,x^3 + 36\,x^2y + 54\,xy^2 + 27\,y^3$.

Ex. 2. $(a + b + c)^3 = (\overline{a + b} + c)^3 = (a + b)^3 + 3\,(a + b)^2c + 3\,(a + b)\,c^2 + c^3 = (a^3 + 3\,a^2b + 3\,ab^2 + b^3) + 3\,(a^2 + 2\,ab + b^2)\,c + 3\,(a + b)\,c^2 + c^3 = a^3 + b^3 + c^3 + 3\,a^2b + 3\,a^2c + 3\,ab^2 + 3\,b^2c + 3\,ac^2 + 3\,bc^2 + 6\,abc$.

(In the following examples the above law may be *assumed* as *generally* true.)

Ex. X.

1. Find the values of $(a^2b^3)^2$, $(-3\,ab^3)^3$, $(2\,a^2bc)^4$, $(-x^3yz^2)^3$

Expand—

2. $(a + 3b)^4$, $(2a + b)^4$, $(a - b)^5$, $(3a - 4b)^3$.

3. $(2m + 1)^4$, $(5x + 2)^3$, $(3a - 4c)^4$, $(-a - b)^3$.

4. $(x^2 + x + 1)^2$, $(3a - b + 4c - d)^2$, $(a + 2b - c)^2$ $(3a + 3b + 3c)^2$.

5. $(1 + x - x^2)^3$, $(ax + by + cz)^3$, $\left\{(a+b)x-(c+d)y\right\}^3$.

6. $(1 + x)^7$, $(1 + x + x^2)^4$, $(a + bx + cx^2)^4$.

7. $(ax - by)^8$, $(3x + y)^3 (3x - y)^3$, $(x^2 + xy + y^2)^3 (x - y)^3$.

8. $(x^4 + x^2y^2 + y^4)^2 (x + y)^2 (x - y)^2$,

$\left\{(a + b)^2 - 4ab\right\}^3 \left\{(a - b)^2 + 4ab\right\}^3$.

Simplify—

9. $(a + b + c)^3 - 3(a + b + c)^2 c + 3(a + b + c)c^2 - c^3$.

10. $(a - b)^3 + 3(a - b)^2 (b - c) + 3(a - b)(b - c)^2 + (b - c)^3$.

11. $(1 + x + 3x^2 + 3x^3)^3 + (1 - x + 3x^2 - 3x^3)^3$.

12. $\left\{(x + y)^3 - (x^3 + y^3)\right\}^3 - 27x^3y^3(x^3 + y^3)$.

Evolution.

34. **Evolution** is the operation by which we obtain the *roots* of quantities.

Since the *square* or *second power* of a^3 is $(a^3)^2$ or a^6, we call (Art. 17) a^3 the *square root* or the *second root* of a^6.

And so, since the *cube* or *third power* of a^2 is $(a^2)^3$ or a^6, we call a^2 the *cube root* or the *third root* of a^6.

So, generally, since the nth power of $a^m = (a^m)^n = a^{mn}$, we call a^m the nth root of a^{mn}.

Thus, we have $\sqrt{a^6} = a^3$, $\sqrt[3]{a^6} = a^2$, $\sqrt[n]{a^{mn}} = a^m$.

Hence, in the case of quantities consisting of a single letter with a given exponent, when the given exponent contains as a factor the *number* indicating the root, we must divide the *given exponent* by this number, the quotient being the exponent of the root.

(We shall see farther on that this rule holds when the given exponent is not so divisible, the root in this case being called a surd).

35. Since the product of an *even* number of *negative factors* must give a *positive* result, and the product of an *odd* number of *negative* factors a *negative* result, it follows that—

I. When the root is indicated by an *even* number—

1. The root of a *positive* quantity may be written either with a + or − sign.

Thus, $\sqrt[2]{9a^2} = \pm 3a$, $\sqrt[4]{16b^4} = \pm 2b$.

2. The root of a *negative* quantity is impossible.

Thus, $\sqrt[2]{-a^2}$, $\sqrt[4]{-a^4b^{12}}$, &c., are impossible quantities.

II. When the root is indicated by an *odd* number, the root has always the sign of the given quantity.

Thus, $\sqrt[3]{-27b^6} = -3b^2$, $\sqrt[5]{32x^{10}y^5} = 2x^2y$.

(It may be remarked that the theory of *impossible quantities* forms an important branch of Algebra, which the student cannot yet enter upon. According to that theory, all quantities have as many roots as the number indicating the root.)

Square Root.

36. We shall now develop the method of finding the square root of a given quantity.

Ex. 1. Find the square root of $a^2 + 2ab + b^2$.

We know that $a^2 + 2ab + b^2 = (a + b)^2$.

Hence, $a + b$ is the square root of $a^2 + 2ab + b^2$ or $a^2 + (2a + b)b$.

Now, it is evident that the first term a of the root is the square root of the first term a^2 of the given quantity; and if this term be subtracted, there remains $2ab + b^2$, from which to determine b the second term of the root. Now, b is contained in $2ab + b^2$ or $(2a + b)b$ exactly $(2a + b)$ times. Hence it follows that the second term of the root is found by dividing the remainder by twice the first term of the root, and, if we wish to arrange our work in a way similar to long division, it is evident that we first take for our divisor $2a + b$, that is, twice the first term of the root added to the second term, which multiplied by b the second term, and subtracted, leaves no remainder.

Thus the whole process may be arranged as follows :—

$$\begin{array}{ll} & a^2 + 2ab + b^2(a + b \\ & a^2 \\ \hline 2a + b & \quad 2ab + b^2 \\ & \quad 2ab + b^2 \\ \end{array}$$

Ex. 2. Find the square root of $a^2 + 2ab + b^2 + 2ac + 2bc + c^2$.

Now we know (Art. 26) that—

$$a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \text{ or } (a + b)^2 + 2(a + b)c + c^2 = (a + b + c)^2.$$

And if we compare the form $(a + b)^2 + 2(a + b)c + c^2$ with the form $a^2 + 2ab + b^2$, it is evident that, having obtained as in the last example the first two terms $a + b$, we shall by continuing the process obtain the third term. Thus—

$$\begin{array}{ll} & a^2 + 2ab + b^2 + 2ac + 2bc + c^2(a + b + c \\ & a^2 \\ \hline 2a + b & 2ab + b^2 \\ & 2ab + b^2 \\ \hline 2a + 2b + c & \qquad 2ac + 2bc + c^2 \\ & \qquad 2ac + 2bc + c^2 \\ \hline \end{array}$$

We may deduce from the above examples the following general rule :—

Rule.

1. Arrange the terms of the given quantity according to the ascending or descending powers of some letter, and take the square root of the first term for the first term of the quotient.

2. Subtract the square of the quotient, and bring down the next two terms of the given quantity.

3. Double the quotient, and place the result as a *trial divisor;* then, dividing the first of the terms brought down by this trial divisor to obtain the second term of the root, add the quotient so obtained to the first term of the root, and also to the trial divisor, to obtain a *complete* divisor.

4. Multiply the complete divisor by the second term of the root, and subtract the product, as in long division, from the terms brought down.

5. If there be any remainder or more terms to bring down, double the whole quotient for a trial divisor, and divide the

remainder by the new trial divisor, to obtain the third term of the root; and so on.

Ex. 3. Find the square root of $1 - 4x + 10x^2 - 12x^3 + 9x^4$. The terms are here arranged according to the ascending powers of x. Then proceeding according to rule, we have—

$$\begin{array}{ll}
 & 1 - 4x + 10x^2 - 12x^3 + 9x^4 \,(1 - 2x + 3x^2 \\
 & 1 \\
\cline{2-2}
2 - 2x & \quad - 4x + 10x^2 \\
 & \quad - 4x + 4x^2 \\
\cline{2-2}
2 - 4x + 3x^2 & \qquad\qquad 6x^2 - 12x^3 + 9x^4 \\
 & \qquad\qquad 6x^2 - 12x^3 + 9x^4 \\
\cline{2-2}
\end{array}$$

The student will observe that twice the quotient is most easily obtained by bringing down the previous complete divisor with its last term doubled.

Square Root of Numerical Quantities.

37. It is easy to apply the above method to numerical quantities.

Since $1^2 = 1$, $10^2 = 100$, $100^2 = 10{,}000$, $1{,}000^2 = 1{,}000{,}000$, &c it is evident that the square roots of numbers having less than three figures must contain one figure only;

That those having not less than three and less than five must contain two figures and two only;

Those having not less than five and less than seven must contain three figures and three only; and so on.

Hence it follows that, if a dot be placed over the units' figure, and over every alternate figure to the left, the number of dots will give the number of figures in the square root.

Thus, the square roots of the numbers $1\dot{4}1\dot{3}7\dot{6}$ and $\dot{1}5\dot{2}2\dot{7}5\dot{6}$ have three and four figures respectively.

In the number $1\dot{4}1\dot{3}7\dot{6}$ we call 14, 13, 76 respectively the first, second, and third periods. So in the number $\dot{1}5\dot{2}2\dot{7}5\dot{6}$, the first, second, third, and fourth periods are respectively 1, 52, 27, 56.

It is evident that the number of periods correspond to the number of figures in the square root, and it will be seen that the figures of each period are used in the operation for the corresponding figure of the root.

Ex. Find the square root of 565504.

Pointing off the number, we find the first period to be 56. Now, the greatest square in 56 is 49, and the square next greater is 64. Hence the number lies between the square of 700 and 800; and, following the algebraical method, 700 will be the first term of the root.

The operation will stand thus—

$$\begin{array}{lrl}
 & 5\dot{6}5\dot{5}0\dot{4} & (\overset{a}{700}+\overset{b}{50}+\overset{c}{2}=752\\
 & 490000 & = a^2\\ \cline{2-2}
2a+b=1400+50=1450 & 75504 & \\
 & 72500 & = 2ab+b^2\\ \cline{2-2}
2a+2b+c=1400+100+2=1502 & 3004 & \\
 & 3004 & = 2ac+2bc+c^2\\ \cline{2-2}
\end{array}$$

Or, omitting the useless ciphers, and bringing down one period of figures at a time, the operation will stand thus—

$$\begin{array}{lr}
 & 5\dot{6}5\dot{5}0\dot{4}(752\\
 & 49\\ \cline{2-2}
145 & 755\\
 & 725\\ \cline{2-2}
1502 & 3004\\
 & 3004\\ \cline{2-2}
\end{array}$$

Ex. 1. Find the square root of 6091024.

$$\begin{array}{lr}
 & \dot{6}0\dot{9}1\dot{0}2\dot{4}(2468\\
 & 4\\ \cline{2-2}
44 & 209\\
 & 176\\ \cline{2-2}
486 & 3310\\
 & 2916\\ \cline{2-2}
4928 & 39424\\
 & 39421\\ \cline{2-2}
\end{array}$$

Ex. 2. Find the square root of 83521.

$$\begin{array}{lr}
 & \dot{8}3\dot{5}2\dot{1}(289\\
 & 4\\ \cline{2-2}
48 & 435\\
 & 384\\ \cline{2-2}
569 & 5121\\
 & 5121\\ \cline{2-2}
\end{array}$$

It will be observed that the second remainder, 51, is greater than the previous complete divisor, 48, and it might be supposed, therefore, that the second figure in the root should be 9 instead of 8.

Now, the square of $(a + 1)$ exceeds the square of a by $2a + 1$.

Thus, $(a + 1)^2 - a^2 = (a^2 + 2a + 1) - a^2 = 2a + 1$.

Hence it follows that, so long as any remainder is less than twice the corresponding number in the root + 1, we may be certain that we have taken the figure of the root sufficiently large.

Thus, since the remainder is less than $28 \times 2 + 1$ or 57, we may be certain that 8 is the correct figure and not 9.

Square Root of a Decimal.

38. It is evident that the *square* of any number containing one, two, three, &c., decimal figures, will contain *two, four, six,* &c., decimal figures respectively; and, hence, conversely, every decimal considered as a square must contain an even number of decimal figures, and its square root must contain half this even number of figures. It will then be necessary to add a cipher when the given number of decimal figures is odd.

Further, since decimals and integers follow the same system of notation, it is evident that if a dot be placed over the *units'* figure of the given number, the pointing off may be performed with regard to the integral part exactly as in integers, there being no necessity to point off the decimals, only taking care to bring them down in pairs, and putting a decimal point in the quotient when the first pair is brought down.

And again, if an integer be given which is not a perfect square, we may, by affixing to the right of it a decimal point and an even number of ciphers, gradually approximate to the square root as nearly as we please.

Ex. 3. Find the square root of 1·8225.

	$\dot{1}\cdot 8225(1\cdot 35$
	1
23	82
	69
265	1325
	1325

Ex. 4. Find the square root of 247 to four places of decimals.

Instead of adding a decimal point and eight ciphers to the right of the given number, we will proceed in the ordinary way till we arrive at a remainder. Then putting a decimal point in the quotient, we shall add two ciphers to this and each successive remainder.

	$\dot{2}4\dot{7}(15\cdot 7162$
	1
25	147
	125
307	2200
	2149
3141	5100
	3141
31426	195900
	188556
314322	734400
	628644
	105756

39. It will be shown hereafter that *when* n + 1 *figures of a square root have been obtained by the ordinary method,* n *figures more may be obtained by dividing the remainder by the number formed by taking twice the quotient already obtained, provided that the whole number of figures in the root is* 2 n + 1.

Ex. Find the square root of 29 to six places of decimals.

The square root required will evidently contain *seven* figures. We shall therefore find the first *four* figures by the ordinary method, and the other *three* by the above method. Thus—

	29(5·385164	
	25	
103	400	
	309	
1068	9100	
	8544	
10765	55600	
	53825	
10770	17750	then by division,
	10770	
	69800	
	64620	
	51800	
	43080	
Ans. 5·385164.	8720	

Cube Root.

40. We will next develop the method of finding the cube root of a quantity.

Ex. 1. Find the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

We know (Art. 33) that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Hence $a + b$ is the cube root of $a^3 + 3a^2b + 3ab^3 + b^3$.

We see then that, the quantity being arranged according to the powers of a, the first term a of the cube root is the cube root of the first term of the given quantity; and if this term a^3 be subtracted, there remains $3a^2b + 3ab^2 + b^3$.

We see again that if this remainder be divided by $3a^2$, its first term gives b the second term of the root, and, further, if it be divided by b, we get $3a^2 + 3ab + b^2$ as a quotient. If we wish therefore to arrange the whole process in a way similar to ordinary division, it is evident that we must write as a divisor $3a^2 + 3ab + b^2$, in order that after multiplication by the quotient figure b we may obtain a quantity which when subtracted shall leave no remainder. The operation will then stand thus—

$$
\begin{array}{ll}
 & a^3 + 3a^2b + 3ab^2 + b^3(a + b \\
 & a^3 \\
\cline{2-2}
3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 \\
 & 3a^2b + 3ab^2 + b^3 \\
\cline{2-2}
\end{array}
$$

We call $3a^2$ the *trial divisor*, because by means of it we search for the second term of the cube root. Having obtained this second term, we then form *the complete divisor* $3a^2 + 3ab + b^2$.

Ex. 2. Find the cube root of $8x^3 - 36x^2y + 54xy^2 - 27y^3$.

$$
\begin{array}{ll}
 & 8x^3 - 36x^2y + 54xy^2 - 27y^3(2x - 3y \\
 & 8x^3 \\
\cline{2-2}
12x^2 - 18xy + 9y^2 & -36x^2y + 54xy^2 - 27y^3 \\
 & -36x^2y + 54xy^2 - 27y^3 \\
\cline{2-2}
\end{array}
$$

Explanation.—We find the cube root of $8x^3$ to be $2x$. This is then the first term of the quotient, and corresponds to a in the previous example. We now require $3a^2$ for a trial divisor. This, of course, $= 3(2x)^2 = 12x^2$. Subtracting the first term of the given quantity and dividing the first term of the remainder by this trial divisor, we obtain $-3y$ for the quotient. This forms the second term of the root, and corresponds to b in the last example. We now easily obtain $3ab$ and b^2.

Thus, $3ab = 3(2x)(-3y) = -18xy$, and $b^2 = (-3y)^2 = 9y^2$.

Hence, the complete trial divisor, corresponding to $3a^2 + 3ab + b^2$ in the last example, $= 12x^2 - 18xy + 9y^2$.

Multiplying now by $-3y$ the quotient, we obtain $-36x^2y + 54xy^2 - 27y^3$, which subtracted leaves no remainder.

Hence, $2x - 3y$ is the cube root.

Ex. 3. Find the cube root of—

$$x^6 + 6x^5y - 40x^3y^3 + 96xy^5 - 64y^6.$$

$$\begin{array}{rl}
 & x^6 + 6x^5y - 40x^3y^3 + 96xy^5 - 64y^6 \left\{ \begin{array}{l} x^2 + 2xy \\ -4y^2 \end{array} \right. \\
 & x^6 \\ \hline
3x^4 + 6x^3y + 4x^2y^2 & 6x^5y \qquad\qquad - 40x^3y^3 \\
 & 6x^5y + 12x^4y^2 + 8x^3y^3 \\ \hline
 & -12x^4y^2 - 48x^3y^3 + 96xy^5 - 64y^6
\end{array}$$

$$\begin{array}{rl}
3(x^2 + 2xy)^2 = & 3x^4 + 12x^5y + 12x^2y^2 \\
3(x^2 + 2xy)(-4y^2) = & \qquad\qquad -12x^2y^2 - 24xy^3 \\
(-4y^2)^2 = & \qquad\qquad\qquad\qquad\qquad 16y^4 \\ \hline
 & 3x^4 + 12x^3y \qquad -24xy^3 + 16y^4 \qquad -12x^4y^2 - 48x^3y^3 + 96xy^5 - 64y^6 \\ \hline
\end{array}$$

It will be evident that the first two terms are obtained exactly as in Exs. 1 and 2. To obtain the next term we treat the terms already found as corresponding to a in Ex. 1. Then we obtain the trial divisor $3(x^2 + 2xy)^2$ corresponding to $3a^2$, and afterwards having divided the first term of the remainder by the first term of the trial divisor in order to get the term corresponding to b, we obtain in order $3(x^2 + 2xy)(-4y^2)$ corresponding to $3ab$, then $(-4y^2)^2$ corresponding to b^2, and lastly, by addition, $3x^4 + 12x^3y - 24xy^3 + 16y^4$, being the complete divisor, corresponding to $3a^2 + 3ab + b^2$ of Ex. 1. We then conclude the operation as before.

Cube Root of Numerical Quantities.

41. We shall now apply the above method to numerical quantities. It may be shown, as in Art. 34, that, if we place a dot over the *units' figure*, and over every *third* figure to the left, the number of periods so formed will be the number of figures in the root.

Ex. 1. Find the cube root of 262144.

$$\begin{array}{rcrr} & & & 26\dot{2}14\dot{4}(\overset{a}{60} + \overset{b}{4} = 64 \\ & & & 216000 \\ 3a^2 & = & 10800 & \overline{46144} \\ 3ab & = & 720 & \\ b^2 & = & 16 & \\ 3a^2 + 3ab + b^2 & = & \overline{11536} & \underline{46144} \end{array}$$

Explanation.—Pointing off the given number, we find the first period to be 262, and that the cube root consists of two figures. Now, the greatest perfect cube in 262 is 216, which is the cube of 6. Hence, the given number lies between the cubes of 60 and 70; and following the algebraical method, 60 will be the first term of the cube. This, we see, corresponds to a in the algebraical method.

We first then subtract the cube of a—viz., 216000, which leaves as a remainder 46144.

We now write down $3a^2$ or $3(60)^2 = 10800$, which is the trial divisor for determining b. Dividing then by this value of $3a^2$, we find $b = 4$, which is the second term of the cube root. We next obtain $3ab = 3(60)(4) = 720$, and $b^2 = 4^2 = 16$, and so by addition we get $3a^2 + 3ab + b^2 = 11536$, which is the complete divisor. Multiplying this then by the quotient figure, we subtract the product, and, there being no remainder, we find the cube root to be 64.

We may omit the useless ciphers in the above operation, if, remembering the local value of figures when numbers are expressed in ordinary notation, we take care to place the right-hand figure of the value of $3ab$ one place to the right of the corresponding figure of the value of $3a^2$; and also to place the right hand figure of b^2 one figure further still to the right.

The operation will then stand thus—

$$\begin{array}{lcrr} & & & 26\dot{2}14\dot{4}(64 \\ & & & 216 \\ 3 \times 6^2 & = & 108 & \overline{46144} \\ 3 \times 6 \times 4 & = & 72 & \\ 4^2 & = & 16 & \\ & & \overline{11536} & \underline{46144} \end{array}$$

Ex. 2. Find the cube root of 102503232.

$$\begin{array}{rr@{\qquad\qquad}r}
 & & 10\dot{2}50\dot{3}23\dot{2}(468 \\
 & & 64 \\ \cline{3-3}
3 \times 4^2 = & 48 & 38503 \\
\begin{array}{r} 3 \times 4 \times 6 = \\ 6^2 = \\ \\ \\ \end{array} & \left.\begin{array}{r} 72 \\ 36 \\ \hline 5556 \\ 36 \end{array}\right\} & \begin{array}{r} \\ \\ 33336 \\ \hline 5167232 \end{array} \\ \cline{2-2}
3 \times 46^2 = & 6348 & \\
3 \times 46 \times 8 = & 1104 & \\
8^2 = & 64 & \\ \cline{2-2}
 & 645904 & 5167232 \\ \cline{2-3}
\end{array}$$

Explanation.—The first two figures of the root are obtained as in Ex. 1. We then treat the number they form, viz., 46, as corresponding to a in the algebraical model, omitting useless ciphers. Obtaining then $3a^2$ or $3 \times 46^2 = 6348$, we find 8 to be the next figure of the root. Then writing under this, $3ab$ or $3 \times 46 \times 8 = 1104$, and afterwards b^2 or $8^2 = 64$, taking care as to the positions of the right-hand figures, and adding, we get 645904 as the complete divisor. Then as before.

Remark.—It is unnecessary to be at the trouble to find the value of 3×46^2 by ordinary multiplication. For referring to the algebraical model, and writing here the successive terms of the complete divisor, and adding, we have—

$$\begin{array}{rl}
 & 3a^2 \\
 & \left.\begin{array}{r} 3ab \qquad \\ b^2 \\ \hline 3a^2 + 3ab + b^2 \\ b^2 \end{array}\right\} \\
\end{array}$$

Sum = $3a^2 + 3ab + b^2$. If we now again write down b^2 under this sum, and then add up *the last four lines*, we get—

$$3a^2 + 6ab + 3b^2, \text{ or } 3(a^2 + 2ab + b^2) = 3(a + b)^2.$$

This is three times the square of the first two terms of the root.

It therefore follows that, if, as in the above example, after completing the operation for finding the first two figures of the cube root, we write under the complete divisor just obtained the value of the square of the second figure, and then add together the last four lines thus obtained, we get three times the square of the quotient for a partial divisor by which to determine the next figure of the root.

The four lines to be added are in the above example bracketed. This method will be found to materially shorten the work, for it may be similarly applied to find the trial divisor when the cube root consists of any number of figures

Cube Root of a Decimal.

42. We know that the *cube* of any number containing one, two, three, &c., decimal figures will contain three, six, nine, &c., decimal figures respectively, and hence, conversely, every decimal considered as a cube must contain a number of decimal figures which is a multiple of *three*, and the number of decimal figures in the *cube root* must be *one-third* of the number contained in the given *cube*. It will then be necessary to add ciphers when the given number of decimal figures is not a multiple of 3.

And by continuing the reasoning of Art. 37, if a dot be placed over the units' figure and over every third figure to the left, it will be sufficient to bring down the decimal figures *three* at a time, putting a decimal point in the quotient when the first three are brought down.

And further, if an integer be given which is not a perfect cube, we may proceed in the ordinary way till we arrive at a remainder, and then, putting a decimal point in the quotient, by affixing *three* ciphers to this and each successive remainder, approximate to the cube root as nearly as we please.

Ex. 3. Find the cube root of 395·446904.

$$\begin{array}{rrl}
 & & 39\dot{5}\cdot446904(7\cdot34 \\
 & & \underline{343} \\
3\times7^2 = & 147 & 52446 \\
\begin{array}{r}3\times7\times3=\\3^2=\\ \\ \\ \end{array} & \left.\begin{array}{r}63\\ \underline{9}\\15339\\9\end{array}\right\} & \begin{array}{l} \\ \\ \underline{\dot{4}6017}\\ 6429904\end{array} \\
3\times73^2= & \overline{15987} & \\
3\times73\times4= & 876 & \\
4^2= & \underline{16} & \\
 & 1607476 & \underline{6429904}
\end{array}$$

43. We shall show farther on that *when* n + 2 *figures of a cube root have been obtained by the ordinary method,* n *figures more may be obtained by dividing the remainder by the next trial divisor, provided that the whole number of figures in the root is* 2 n + 2.

We may apply this principle with advantage when we require the cube root of number to a given number of decimals.

Ex. Find the cube root of 87 to five places of decimals.

The required cube root will evidently contain 6 figures, and since 6 here corresponds to $2n + 2$ above, it is evident that $n = 2$. Hence, we shall find 4 (that is, $n + 2$) figures by the ordinary method, and then 2 more by division.

The operation will stand thus—

$$\begin{array}{rcrl}
 & & & 8\dot{7}(4{\cdot}43104 \\
 & & & \underline{64} \\
3 \times 4^2 & = & 48 & 23000 \\
3 \times 4 \times 4 & = & 48 & \\
4^2 & = & \underline{16} & \\
 & & 5296 & \underline{21184} \\
 & & \underline{16} & 1816000 \\
3 \times 44^2 & = & 5808 & \\
3 \times 44 \times 3 & = & 396 & \\
3^2 & = & \underline{9} & \\
 & & 584769 & \underline{1754307} \\
 & & \underline{9} & 61693000 \\
3 \times 443^2 & = & 588747 & \\
3 \times 443 \times 1 & = & 1329 & \\
1^2 & = & \underline{1} & \\
 & & 58887991 & \underline{58887991} \\
 & & \underline{1} & 280500900 \\
 & & 58901283 & \underline{235605132} \\
 & & & 44895768
\end{array}$$

Ans. 4·43104.

Ex. XI.

Find the square roots of—

1. $4x^2y^4z^6$, $16a^4y^4$, $x^4 + 2a^2x^2 + a^4$.
2. $4x^6 - 12x^5y + 29x^4y^2 - 30x^3y^3 + 25x^2y^4$.

3. $25a^4 - 30a^3b + 19a^2b^2 - 6ab^3 + b^4$.

4. $1 - 4x + 10x^2 - 20x^3 + 25x^4 - 24x^5 + 16x^6$.

5. $a^2 + 2abx + (2ac + b^2)x^2 + 2(ad + bc)x^3 + (2bd + c^2)x^4 + 2cdx^5 + d^2x^6$.

6. $a^4x^{2n} - 6a^3x^{2n-1} + 17a^2x^{2n-2} - 24ax^{2n-3} + 16x^{2n-4}$.

7. $x^2 + 2 + x^{-2}$, $a^2x^{-2} - 2 + a^{-2}x^2$.

8. $9x^{2m} - 3a^2x^m + 25a^2 - 30ax^m + \frac{a^4}{4} + 5a^4$.

Find the square roots of—

9. 1296, 6241, 42849, 83521.

10. 10650·24, ·000576, $\cdot\dot{1}$, $\frac{48}{243}$.

11. $\sqrt{17}$, $\sqrt{1\cdot5}$, $\frac{2}{\sqrt{5}+1}$, $\frac{\sqrt{3}+1}{\sqrt{3}-1}$.

Give the values correct to four places of decimals of—

12. $\frac{\frac{5}{8} \text{ of } \cdot31416}{\sqrt{93}}$, $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}} + \frac{\sqrt{10}-2}{\sqrt{10}+2}$, $3\cdot1416$ of $\sqrt{\frac{40}{32\cdot16}}$.

Find the cube roots of—

13. $8a^3b^6y^{12}$, $125x^{12}y^3$, $a^3 + 6a^2b + 12ab^2 + 8b^3$.

14. $x^{12} + 9x^{10} + 6x^8 - 99x^6 - 42x^4 + 441x^2 - 343$.

15. $x^3 + 3x^2y + 3xy^2 + y^3 - 6cxy - 3cx^2 - 3cy^2 + 3c^2x + 3c^2y - c^3$.

16. $x^3 + x^{-3} + 3(x + x^{-1})$, $x^3y^{-3} + 3x^2y^{-2} + 3xy^{-1} + 1$.

Find the cube roots of—

17. 5849513501832, 1371·330631.

18. 20·346417; $\cdot\dot{0}3\dot{7}$, $\frac{108}{1372}$.

Give the value of the following correct to four places of decimals:—

19. $\frac{\sqrt[3]{5\cdot12} + \sqrt[3]{\cdot03375}}{\sqrt[3]{80} - \sqrt[3]{\cdot01}}$, $\frac{1}{\sqrt[3]{4} + \sqrt[3]{2} + 1}$.

20. $\frac{\sqrt[3]{\cdot625} + \sqrt[3]{\cdot04}}{\sqrt{\cdot05} + \sqrt{\cdot04}}$ of $\frac{\sqrt{5}+2}{7}$.

21. $(\sqrt{7} + 2)(\sqrt{7} - 1)$, $(5 + \sqrt{3})(4 + \sqrt{12})$.

22. $\sqrt{11 + 6\sqrt{2}}$, $\sqrt[3]{6 + 15\sqrt{3}}$.

23. $\sqrt{\frac{3}{4} - \frac{6 + 2\sqrt{5}}{16}} \cdot \frac{\sqrt{5} + 1}{4}$.

24. $a^3(b - c) - b^3(a - c) + c^3(a - b)$, where $a = \sqrt{1 \cdot 2}$, $b = -\sqrt{\cdot 3}$, and $c = -\sqrt[6]{\cdot 027}$.

CHAPTER IV.

GREATEST COMMON MEASURE AND LEAST COMMON MULTIPLE.

Greatest Common Measure.

44. In Arithmetic (page 24) we defined the G.C.M. of two or more numbers as their highest common factor. In Algebra the same definition will suffice, provided we understand by the term *highest common factor*, *the factor of highest dimensions* (Art. 18). This, it need hardly be remarked, does not necessarily correspond to the factor of highest numerical value.

45. To find the G.C.M. of two quantities.

Rule.—Let A and B be the quantities, of which A is not of lower dimensions than B. Divide A by B, until a remainder is obtained of lower dimensions than B. Take this remainder as a new divisor, and the preceding divisor A as a new dividend, and divide till a remainder is again obtained of lower dimensions than the divisor; and so on. The last divisor is the G.C.M.

Before giving the general theory of the G.C.M. we shall work out a few examples.

Ex. 1. Find the G.C.M. of $x^2 - 6x - 27$ and $2x^2 - 11x - 63$.

According to the above rule, the operation is as follows:—

$$\begin{array}{r@{\,}l}
x^2 - 6x - 27) & 2x^2 - 11x - 63(2 \\
& \underline{2x^2 - 12x - 54} \\
x - 9) & x^2 - 6x - 27(x + 3 \\
& \underline{x^2 - 9x\phantom{{}-27}} \\
& \phantom{x^2 - {}}3x - 27 \\
& \phantom{x^2 - {}}\underline{3x - 27}
\end{array}$$

∴ The G.C.M. is $x - 9$.

Ex. 2. Find the G.C.M. of $10x^3 + 31x^2 - 63x$ and $14x^3 + 51x^2 - 54x$.

We may tell by inspection that x is a common factor, which we therefore strike out of both, only *taking care to reserve* it. The quantities then become—

$$10x^2 + 31x - 63, \text{ and } 14x^2 + 51x - 54.$$

We may now proceed according to rule, taking the former as divisor. We see, however, that the coefficient of the first term of the dividend is not exactly divisible by the coefficient of the first term of the divisor. Multiply therefore (to avoid fractions) the dividend by such a number as will make it so divisible, viz., by 5. This will not affect the G.C.M., as 5 is not *a factor* of the first expression, viz., $10x^2 + 31x - 63$.

It may as well be here mentioned that the G.C.M. of two quantities cannot be affected by the multiplication or division of *one* of the quantities by any quantity which is not a measure of the other. We shall, for a similar reason, reject certain factors or introduce them into any of the remainders or dividends during the operation. (See Art. 47).

$$\begin{array}{rl} & 14x^2 + 51x - 54 \\ & \underline{5\qquad\qquad\qquad\quad} \\ 10x^2 + 31x - 63) & 70x^2 + 255x - 270(7 \\ & \underline{70x^2 + 217x - 441} \\ & \qquad\quad 38x + 171 \end{array}$$

Rejecting the factor 19 of this remainder, we have—

$$\begin{array}{rl} 2x + 9) & 10x^2 + 31x - 63(5x - 7 \\ & \underline{10x^2 + 45x\qquad\quad} \\ & \qquad - 14x - 63 \\ & \qquad \underline{- 14x - 63} \end{array}$$

Hence, $2x + 9$ is the last divisor, and multiplying this by x, the common measure struck out at the commencement, we find the G.C.M to be $x(2x + 9)$ or $2x^2 + 9x$.

Ex. 3. Find the G.C.M. of $x^6-7x^5-3x^4-5x^3+42x^2-34x-21$, and $x^5-11x^4+25x^3+19x^2-49x-21$.

$$\begin{array}{r}
x^5 - 11x^4 + 25x^3 + 19x^2 - 49x - 21)\,x^6 - 7x^5 - 3x^4 - 5x^3 + 42x^2 - 34x - 21\,(x+4 \\
\underline{x^6 - 11x^5 + 25x^4 + 19x^3 - 49x^2 - 21x} \\
4x^5 - 28x^4 - 24x^3 + 91x^2 - 13x - 21 \\
\underline{4x^5 - 44x^4 + 100x^3 + 76x^2 - 196x - 84} \\
16x^4 - 124x^3 + 15x^2 + 183x + 63
\end{array}$$

Multiplying the preceding divisor by 16, and taking the result for a dividend, we have—

$$\begin{array}{r}
16x^4 - 124x^3 + 15x^2 + 183x + 63)\,16x^5 - 176x^4 + 400x^3 + 304x^2 - 784x - 336\,(x \\
\underline{16x^5 - 124x^4 + 15x^3 + 183x^2 + 63x} \\
-52x^4 + 385x^3 + 121x^2 - 847x - 336 \\
\text{(Multiplying this remainder by 4)} \qquad \underline{4} \\
-208x^4 + 1540x^3 + 484x^2 - 3388x - 1344\,(-13 \\
\underline{-208x^4 + 1612x^3 - 195x^2 - 2379x - 819} \\
-72x^3 + 679x^2 - 1009x - 525 \;-
\end{array}$$

Multiplying the preceding divisor by 9, and taking the result for a dividend, we have—

$$\begin{array}{r}
-72x^3 + 679x^2 - 1009x - 525)\,144x^4 - 1116x^3 + 135x^2 + 1647x + 567\,(-2x \\
\underline{144x^4 - 1358x^3 + 2018x^2 + 1050x} \\
242x^3 - 1883x^2 + 597x + 567 \\
\text{(Multiplying this remainder by 36)} \qquad \underline{36} \\
8712x^3 - 67788x^2 + 21492x + 20412\,(121 \\
\underline{8712x^3 - 82159x^2 + 122089x + 63525} \\
14371x^2 - 100597x - 43113
\end{array}$$

Dividing this remainder by 14371, and taking the quotient for a new divisor, we have—

$$\begin{array}{l}
x^2 - 7x - 3)\ -72x^3 + 679x^2 - 1009x - 525(\ -72x + 175 \\
\qquad\qquad\ \ \underline{-72x^3 + 504x^2 + \ \ 216x\qquad\qquad} \\
\qquad\qquad\qquad\qquad\ 175x^2 - 1225x - 525 \\
\qquad\qquad\qquad\qquad\ \underline{175x^2 - 1225x - 525}
\end{array}$$

$\therefore\ x^2 - 7x - 3$ is the G.C.M.

It will be seen that we have introduced and rejected factors during the operation in order to avoid fractional coefficients. This, as will be seen from the general theory, will not affect the result, provided that no factor thus introduced or rejected is a measure of the corresponding divisor or dividend, as the case may be.

Theory of the Greatest Common Measure.

46. Let A and B be the two algebraical quantities, and the operation as indicated by the rule (Art. 45) be performed. Thus, let A be divided by B, with quotient p and remainder C. Then let B be divided by C, with quotient q, and remainder D. Lastly, let C be divided by D, with quotient r, and remainder zero.

$$\begin{array}{l}
B)A(p \\
\ \ \ \underline{pB} \\
\ \ \ C)B(q \\
\qquad \underline{qC} \\
\qquad D)C(r \\
\qquad\quad \underline{rD} \\
\qquad\quad\ \ 0
\end{array}$$

Then we are required to show that D is the G.C.M. of A and B.

(1.) D is a common measure of A and B.

Now, we have $C = rD$, $B = qC + D$, $A = pB + C$. Hence, D is a measure of C, and therefore of qC. It is therefore a measure of $qC + D$ or B. Hence, also, D is a measure of pB, and since it is also a measure of C, it must be a measure of $pB + C$ or A. But we have shown it to be a measure of B. Hence, D is a common measure of A and B.

(2.) D is the G.C.M. of A and B.

For every measure of A and B will divide $A - pB$ or C; and hence every measure of A and B will divide $B - qC$ or D. Now, D cannot be divided by any quantity higher than D, and, therefore, there cannot exist a measure of A and B higher than D. Hence, D is the G.C.M. of A and B.

47. *A factor which does not contain any factor common to both A and B may be rejected at any stage of the process.*

Let the operation stand thus:—

$$\begin{array}{l} B = mB' \text{ suppose,} \\ B')A(p \\ \quad \underline{pB'} \\ \quad\ C = nC' \text{ suppose,} \end{array} \qquad \begin{array}{l} \\ C')B'(q \\ \quad \underline{qC'} \\ \quad D)C'(r \\ \qquad\ \underline{rD} \\ \qquad\quad 0 \end{array}$$

where neither m nor n contains any factor common to A and B.

It will be an exercise for the student to show that D is the G.C.M. of A and B.

48. *A factor, which has no factor that the divisor has, may be introduced into the dividend at any stage of the process.*

The operation may stand thus—

$$\begin{array}{l} B)mA(p, \text{ where } m \text{ has no factor that } B \text{ has;} \\ \quad \underline{pB} \\ \quad\ C)nB(q, \text{ where } n \text{ has no factor that } C \text{ has;} \\ \qquad \underline{qC} \\ \qquad\ D)C(r \\ \qquad\quad \underline{rD} \\ \qquad\qquad 0 \end{array}$$

As in Arts. 46, 47, it may be easily shown that D is the G.C.M.

Both the above principles are made use of in working out Ex. 3 Art. 45.

49. When a common factor can be found by inspection, it is advisable to strike it out of the given expressions. Then, having found by the ordinary process the G.C.M. of the resulting quantities, we must multiply the G.C.M. so found by the rejected factor.

Thus, $4x$ is common to the quantities $4x^3 - 20x^2 + 24x$, and $4x^3 + 16x^2 - 84x$.

Rejecting it, we get $x^2 - 5x + 6$, and $x^2 + 4x - 21$, whose G.C.M. is easily found to be $x - 2$.

Multiplying by $4x$, we find the required G.C.M. to be $4x^2 - 8x$.

50. By a little ingenuity on the part of the student in breaking up the given expressions into factors, the ordinary and often tedious process of finding the G.C.M. may be avoided. The limits of our space will allow us only one example.

Ex. Find the G.C.M. of $3x^3 + 4x^2 - 10x + 3$, and $15x^3 + 47x^2 + 13x - 12$.

The first expression contains $x - 1$ as a factor (Art. 30), for the sum of its coefficients is *zero*. The other factor may be obtained thus—

$$\begin{aligned} 3x^3 + 4x^2 - 10x + 3 &= 3x^3 - 3x^2 + 7x^3 - 7x - 3x + 3 \\ &= 3x^2(x - 1) + 7x(x - 1) - 3(x - 1) \\ &= (3x^2 + 7x - 3)(x - 1). \end{aligned}$$

Now, $3x^2 + 7x - 3$ is not further resolvable, and $x - 1$ is evidently (Art. 30) not a factor of $15x^3 + 34x^2 + 13x - 12$. It is, therefore, very probable that $3x^2 + 7x - 3$ is the G.C.M. required.

We may test it thus—

$$\begin{aligned} 15x^3 + 47x^2 + 13x - 12 &= 15x^3 + 35x^2 - 15x + 12x^2 + 28x - 12 \\ &= 5x(3x^2 + 7x - 3) + 4(3x^2 + 7x - 3) \\ &= (5x + 4)(3x^2 + 7x - 3). \end{aligned}$$

Hence, $3x^2 + 7x - 3$ is the G.C.M. required.

G.C.M. of Three or More Quantities.

51. The G.C.M. of three or more quantities may be found thus—

RULE.—Find the G.C.M. of any two of the quantities, then the G.C.M. of the G.C.M. so found and a third quantity, and so on. The last found G.C.M. will be the G.C.M. required.

Ex. XII.

Find the G.C.M. of the following—

1. $x^2 - 5x + 6$ and $x^2 + 3x - 18$.
2. $x^3 + 6x^2 + 11x + 6$ and $x^3 + 5x^2 + 7x + 3$.
3. $2x^3 + 10x^2 - 18x - 90$ and $3x^3 + 16x^2 - 26x - 141$.
4. $x^2 + (a + b\ x + ab$ and $x^2 + (a + c)x + ac$.
5. $a^3 - b^3$ and $a^3 + a^2b + ab^2$.

6. $x^3 - 4x + 3$ and $x^3 + 4x^2 - 5$.

7. $4x^3 - 32x^2 + 85x - 75$ and $3x^3 - 15x^2 + 15x + 9$.

8. $9x^2 - 3xy + 2y - 4$ and $6x^4 - 4x^3 - 9xy^2 + 6y^2$.

9. $48x^4 + 8x^3 + 31x^2 + 15x$ and $24x^4 + 22x^3 + 17x^2 + 5x$.

10. $15a^3 + a^2b - 3ab^2 + 2b^3$ and $54a^2b^2 - 24b^4$.

11. $3x^3 - (3c + d + 1)x^2 - (2a + b - 3c - d + 2)x + 2a + b$ and $2x^2 - (a + b + 2)x + a + b$.

12. $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$ and $4x^4 + 2x^3 - 18x^2 + 3x - 5$.

13. $ab + 2a^2 - 3b^2 - 4bc - ac - c^2$ and $9ac + 2a^2 - 5ab + 4c^2 + 8bc - 12b^2$.

14. $e^xx^2 + e^x + x^2 + 1$ and $e^{2x}x^4 - e^{2x} + x^4 - 1$.

15. $ax^4 + (b + c)x^2 - ax - b - c$ and $ex^3 - (f - g)x^2 + (f - e)x - g$.

16. $4x^4 + 2x^3 + 4x^2 + 39x - 9$, $8x^4 + 20x^2 + 51x + 9$, and $2x^4 + x^3 + 3x^2 + 18x$.

17. $ax^3 - (c + 1)x^2 + (c + 1)x - a$, $bx^4 - (b + d)x^3 + (c + d)x^2 - (c + e)x + e$, and $(c + 1)x^5 + (d + 2)x^4 - (d + 1)x^3 - (c + 2)x^2$.

18. $a^3 - b^3 + c^3 + 3abc$ and $a^2 - b^2 + c^2 + 2ac$.

Least Common Multiple.

52. When two or more algebraical expressions are arranged according to the powers of some letter, the expression of lowest dimensions which is divisible by each of the given expressions is called the L.C.M.

53. The L.C.M. of monomials and of expressions whose factors are apparent may be found by inspection.

Ex. 1. Find the L.C.M. of ab, ac, ad, bc, bd, cd.

If we form an expression, whose elementary factors contain each of the elementary factors of the given quantities, we shall evidently have a *common multiple;* and if no elementary factor of this expression is of a higher power than the highest power of the same factor in the given quantities, we shall get the L.C.M.

Hence, the required L.C.M. $= abcd$.

Ex. 2. Find the L.C.M. of—

$(a - b)(b - c), (a - b)(c - a), (b - c)(c - a)$.

Ans. $(a - b)(b - c)(c - a)$.

Ex. 3. Find the L.C.M. of $a(x + 1)$, $b(x^2 - 1)$, $c(x^2 + 2x - 3)$, $d(x^2 + 4x - 3)$. We may write the given expressions thus—

$a(x + 1), b(x + 1)(x - 1)$,

$c(x - 1)(x + 3), d(x + 1)(x + 3)$.

Hence, the required L.C.M. $= abcd(x - 1)(x + 1)(x + 3)$.

Ex. 4. Find the L.C.M. of $a^2 - ax + x^2$, $a^2 + ax + x^2$, $a^3 + x^3$, $a^3 - x^3$.

$$\text{Now (Art. 29) } a^3 + x^3 = (a + x)(a^2 - ax + x^2),$$
$$\text{and } a^3 - x^3 = (a - x)(a^2 + ax + x^2).$$

Hence the required L.C.M.—

$$= (a + x)(a - x)(a^2 + ax + x^2)(a^2 - ax + x^2) = a^6 - x^6.$$

54. *The L.C.M. of two quantities is found by dividing their product by their G.C.M.*

Let a and b be the two quantities, and d the G.C.M.;

And suppose $a = pd$ and $b = qd$.

It is evident that p and q contain no common factor. Hence pq is the L.C.M. of p and q; and, therefore, no expression of lower dimensions than pqd can possibly be divisible by pd and qd.

Hence pqd is the L.C.M. of pd and qd, or of a and b.

Now $pqd = pd \times qd \div d = a \times b \div d$, and hence the rule.

55. To find the L.C.M. of *three* or more quantities.

Rule.—Find the L.C.M. of two of the quantities, then the L.C.M. of the expression thus obtained and a third quantity, and so on. The last expression so found is the L.C.M. required.

We shall prove this rule in the case of *three* quantities.

Let a, b, c be the quantities, and m be the L.C.M. of a and b.

Then the L.C.M. of m and c is the L.C.M. required.

For every common multiple of m and c is a common multiple of a, b, c. And every common multiple of a and b

must contain the m, their least common multiple. Hence, every common multiple of a, b, c must be a common multiple of m and c, and the converse is also true. Hence, the L.C.M. of m and c is the L.C.M. of a, b, c.

Ex. XIII.

Find the L.C.M. of—

1. $axy^2,\ 3a^2x^3y,\ 4a^3y^3,\ 6x^2y^2$.
2. $5a^2b^2,\ 6a^2c^2,\ 4b^2c^2$.
3. $(a - b)(b - c),\ (b - a)(a - c),\ (c - a)(c - b)$.
4. $ax(x + a),\ a^2(x - a),\ x^2 - a^2$.
5. $x^2 + 3x + 2,\ x^2 + 4x + 3,\ x^2 + 5x + 6$.
6. $x^2 - x - 30,\ x^2 - 11x + 30,\ x^2 - 25$.
7. $6x^2 + 37x + 56,\ 8x^2 + 38x + 35,\ 12x^2 + 47x + 40$.
8. $5(x^2 - x + 1),\ 6(x^2 + 1),\ 7(x^3 + 1)$.
9. $x^4 + a^2x^2 + a^4,\ a^2x^2 + a^3x + a^4,\ ax^2 - a^2x + a^3$.
10. $x^2 + (a + b)x + ab,\ x^2 + (a + c)x + ac,\ x^2 + (b + c)x + bc$.
11. $1 - x,\ 1 + x,\ 1 + x^2,\ 1 + x^4,\ 1 + x^8$.
12. $x^3 + 6x^2 + 11x + 6,\ x^3 - 6x^2 - 25x + 150$.
13. $a^3 - 3ab(a - b) - b^3,\ a^3 - b^3,\ a^3 + a^2b + ab^2$.
14. $x^4 - 1,\ 6x^5 + 5x^4 + 8x^3 + 4x^2 + 2x - 1$.
15. $a^4 - 2a^2b^2 + b^4,\ a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,\ a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.
16. $3x^3 - 4x + 1,\ 2x^3 - 7x + 5,\ 4x^4 + 6x^2 + 10x$.
17. $3x^2 + 6x - 24,\ x^3 - 12x + 16,\ 5x^4 - 22x - 36$.
18. $a^3 - ab^2,\ b^3 - a^2b,\ ab^2 - b^3,\ a^2b - a^3$.
19. $3x^4 - 48,\ 5x^2 - 20,\ 3x^2 - 16x + 20$.
20. $x^8 - y^8,\ x^4 - 2x^2y^2 + y^4,\ x^3 + x^2y + xy^2 + y^3$.
21. $x^5 + ax^4 + a^2x^3 + a^3x^2 + a^4x + a^5$ and $x^5 - ax^4 + a^2x^3 - a^3x^2 + a^4x - a^5$.
22. $a^2 + b^2 - c^2 - d^2 + 2ab - 2cd$ and $a^2 - b^2 - c^2 + d^2 + 2ad - 2bc$.

23. $a^3 + b^3 + c^3 - 3abc$ and $(a+b)^2 + 2(a+b)c + c^2$.

24. $(a+b)^2 - (c+d)^2$, $(a+c)^2 - (b-d)^2$, $(a+d)^2 - (b-c)^2$, $(c+d)^2 - (a-b)^2$, $(b+d)^2 - (a-c)^2$, and $(b+c)^2 - (a-d)^2$.

CHAPTER V.

Fractions.

56. It is unnecessary to repeat here the propositions relating to fractions which were proved in Arithmetic, Chap. II. of this work. The student will see that, by substituting general symbols for the particular figures there used, the reasoning will equally hold. We shall work out a few examples to show the method of dealing with them in algebra.

Ex. 1. Simplify the fraction $\dfrac{x^3 - 2x^2 + x - 12}{x^2 + 2x - 15}$.

By inspection (Art. 30) we see that $x - 3$ is a factor of numerator and denominator. We have then—

$$\frac{x^3 - 2x^2 + x - 12}{x^2 + 2x - 15} = \frac{x^2(x-3) + x(x-3) + 4(x-3)}{(x-3)(x+5)}$$

$$= \frac{(x^2 + x + 4)(x-3)}{(x-3)(x+5)} = \frac{x^2 + x + 4}{x+5}. \quad \text{Ans.}$$

Ex. 2. Find the value of $\dfrac{1}{a+b} + \dfrac{1}{a-b} - \dfrac{2a}{a^2+b^2}$.

$$\frac{1}{a+b} + \frac{1}{a-b} - \frac{2a}{a^2+b^2} = \frac{(a-b) + (a+b)}{(a+b)(a-b)} - \frac{2a}{a^2+b^2}$$

$$= \frac{2a}{a^2-b^2} - \frac{2a}{a^2+b^2} = 2a\left(\frac{1}{a^2-b^2} - \frac{1}{a^2+b^2}\right)$$

$$= 2a \,.\, \frac{(a^2+b^2) - (a^2-b^2)}{(a^2-b^2)(a^2+b^2)} = 2a \,.\, \frac{2b^2}{a^4-b^4} = \frac{4ab^2}{a^4-b^4}.$$

Ex. 3. Find the value of $\frac{bc}{(a-b)(a-c)} + \frac{ac}{(b-a)(b-c)} + \frac{ab}{(c-a)(c-b)}$.

The second denominator has a factor, $(b - a)$, which differs from a factor, $(a - b)$, of the first denominator in sign only. We shall therefore change the sign of the second fraction, and also of its first factor. This will not alter its value.

And, similarly, we find that by changing the signs of each of the factors of the third denominator we shall have them in a form corresponding to factors of the first and second denominators. The sign of the third fraction will not be changed, as the sign of the denominator will, on the whole, be unchanged.

The given expression then will stand thus—

$$= \frac{bc}{(a-b)(a-c)} - \frac{ac}{(a-b)(b-c)} + \frac{ab}{(a-c)(b-c)}$$

$$= \frac{bc(b-c) - ac(a-c) + ab(a-b)}{(a-b)(a-c)(b-c)}$$

$$= \frac{bc(b-c) - a^2c + ac^2 + a^2b - ab^2}{(a-b)(a-c)(b-c)}$$, or, re-arranging,

$$= \frac{a^2(b-c) - a(b^2-c^2) + bc(b-c)}{(a-b)(a-c)(b-c)}$$, then, dividing numerator and denominator by $b - c$,

$$= \frac{a^2 - a(b+c) + bc}{(a-b)(a-c)} = \frac{(a-b)(a-c)}{(a-b)(a-c)} = 1.$$

Ex. 4. Simplify—

$$\left\{a^2 - \frac{4a^2x - 3ax^2 + x^3}{a+x}\right\} \times \left\{a^2 + \frac{4a^2x + 3ax^2 + x^3}{a-x}\right\}.$$

The given expression—

$$= \frac{a^2(a+x) - 4a^2x + 3ax^2 - x^3}{a+x} \times \frac{a^2(a-x) + 4a^2x + 3ax^2 + x^3}{a-x}$$

$$= \frac{a^3 + a^2x - 4a^2x + 3ax^2 - x^3}{a+x} \times \frac{a^3 - a^2x + 4a^2x + 3ax^2 + x^3}{a-x}$$

$$= \frac{a^3 - 3a^2x + 3ax^2 - x^3}{a + x} \times \frac{a^3 + 3a^2x + 3ax^2 + x^3}{a - x}$$

$$= \frac{(a - x)^3}{a + x} \times \frac{(a + x)^3}{a - x} = \frac{(a - x)^2}{1} \times \frac{(a + x)^2}{1} = (a^2 - x^2)^2.$$

Ex. 5. Divide—

$$(a+b)\left\{\frac{1}{x - a} + \frac{1}{x + b}\right\} \text{ by } (a - b)\left\{\frac{a}{(x-a)^2} + \frac{b}{(x + b)^2}\right\}.$$

Now—

$$\frac{(a + b)\left\{\frac{1}{x - a} - \frac{1}{x + b}\right\}}{(a - b)\left\{\frac{a}{(x - a)^2} + \frac{b}{(x+b)^2}\right\}} = \frac{(a + b)\,.\,\frac{(x + b) - (x - a)}{(x - a)\,(x + b)}}{(a - b)\,.\,\frac{a(x + b)^2 + b(x - a)^2}{(x - a)^2\,(x + b)^2}},$$

or, reducing—

$$= \frac{(a + b)\,.\,\frac{a + b}{(x - a)\,(x + b)}}{(a - b)\,.\,\frac{(a + b)\,x^2 + ab\,(a + b)}{(x - a)^2\,(x + b)^2}} = \frac{(a + b)\,.\,\frac{1}{1}}{(a - b)\,.\,\frac{x^2 + ab}{(x - a)\,(x + b)}}$$

$$= \frac{(a + b)\,(x - a)\,(x + b)}{(a - b)\,(x^2 + ab)}.$$

Ex. XIV.

Simplify the following expressions:—

1. $\dfrac{x^2 - 5x + 4}{x^2 + 2x - 24}, \dfrac{x^3 - 3x + 2}{x^3 + 4x^2 - 5}.$

2. $\dfrac{6x^2 + 29x + 35}{14x^2 + 39x + 10}, \dfrac{2x^3 + 7x - 9}{5x^3 - 3x^2 - 4x + 2}.$

3. $\dfrac{a^4 - 2a^2b^2 + b^4}{a^3 - 4a^2b + 4ab^2 - b^3}, \dfrac{24a^3 - 28a^2b + 6ab^2 - 7b^3}{6a^2 + 11ab - 21b^2}.$

4. $\dfrac{x^2(y^2 - z^2) - xy(2y^2 + yz - z^2) + y^3(y + z)}{x^2(y + z)^2 - xy(2y^2 + 3yz + z^2) + y^3(y + z)}, \dfrac{x^4 + x^2y^2 + y^4}{x^6 - y^6}.$

5. $\dfrac{1}{a + b} + \dfrac{1}{a - b}, \dfrac{1}{a - b} - \dfrac{1}{a + b}.$

6. $\dfrac{a}{a+b} + \dfrac{b}{a-b} - \dfrac{ab}{ab-b^2} + \dfrac{ab}{a^2+ab}.$

7. $\dfrac{1}{4(x-1)} - \dfrac{x-1}{4(x^2+1)} - \dfrac{x-1}{2(x^2+1)^2}.$

8. $\dfrac{5}{x+1} - \dfrac{2}{(x+2)^2} - \dfrac{18}{5(x+2)} - \dfrac{7x+1}{5(x^2+1)}.$

9. $\dfrac{11}{x+1} - \dfrac{11}{x+3} - \dfrac{14}{(x+1)^2}.$

10. $\dfrac{8}{(x-1)^4} - \dfrac{4}{(x-1)^3} - \dfrac{2}{(x-1)^2} + \dfrac{16x+14}{3(x^2-1)} - \dfrac{16x-8}{3(x^2-x+1)}.$

11. $\dfrac{1}{(a-b)(a-c)} + \dfrac{1}{(b-a)(b-c)} + \dfrac{1}{(c-a)(c-b)}.$

12. $\dfrac{a}{(a-b)(a-c)} + \dfrac{b}{(b-a)(b-c)} + \dfrac{c}{(c-a)(c-b)}.$

13. $\dfrac{a^2}{(a-b)(a-c)} + \dfrac{b^2}{(b-a)(b-c)} + \dfrac{c^2}{(c-a)(c-b)}.$

14. $\dfrac{a^3}{(a-b)(a-c)} + \dfrac{b^3}{(b-a)(b-c)} + \dfrac{c^3}{(c-a)(c-b)}.$

15. $\dfrac{1}{\left(\dfrac{a}{b}-1\right)\left(\dfrac{a}{c}-1\right)} + \dfrac{1}{\left(\dfrac{b}{a}-1\right)\left(\dfrac{b}{c}-1\right)} + \dfrac{1}{\left(\dfrac{c}{a}-1\right)\left(\dfrac{c}{b}-1\right)}.$

16. $\dfrac{(a+b)^2+(b-c)^2+(a+c)^2}{(a+b)(b-c)(a+c)} - \dfrac{2}{a+c} - \dfrac{2}{b-c} + \dfrac{2}{a+b}.$

17. $\left\{\dfrac{1}{a+x} + \dfrac{1}{a-x} + \dfrac{2a}{a^2+x^2}\right\} \cdot \left\{\dfrac{1}{a+x} - \dfrac{1}{a-x} - \dfrac{2x}{a^2+x^2}\right\}.$

18. $\left\{\dfrac{a^2+b^2}{a^2-b^2} + \dfrac{a-b}{a+b}\right\} \div \left\{2 + \dfrac{a^2+b^2}{a^2-b^2} - \dfrac{a-b}{a+b}\right\}.$

19. $\left\{1 - \dfrac{2x^2}{a^2} + \dfrac{2x^4}{a^2(a^2+x^2)}\right\} \times \left\{\dfrac{2a^4}{x^2(a^2-x^2)} - \dfrac{2a^2}{x^2} - 1\right\}.$

20. $\dfrac{a}{c} + \dfrac{b}{a} + \dfrac{c}{b} - \dfrac{a^2\left(\dfrac{b}{c}+1\right) + b^2\left(\dfrac{c}{a}+1\right) + c^2\left(\dfrac{a}{b}+1\right)}{ab+bc+ca}.$

21. $$\left\{\left(\frac{x+y}{x-y}\right)^2+1\right\}\cdot\left\{\left(\frac{x+z}{x-z}\right)^2+1\right\}\cdot\left\{\left(\frac{y+z}{y-z}\right)^2+1\right\}\times\frac{x^2(y-z)+y^2(z-x)+z^2(x-y)}{x^4y^2+x^2y^4+x^4z^2+x^2z^4+y^4z^2+y^2z^4+2x^2y^2z^2}.$$

22. $$\left\{\frac{1-x^2}{1-x^3}+\frac{1-x}{1-x+x^2}\right\}\div\left\{\frac{1+x}{1+x+x^2}-\frac{1-x^2}{1+x^3}\right\}.$$

23. $$\frac{a^2}{(a+b)(a+c)(x-a)}+\frac{b^2}{(a+b)(b-c)(x+b)}-\frac{c^2}{(a+c)(b-c)(x+c)}.$$

24. $$\left\{\frac{9x^2}{a^2}-\frac{24x}{a}+16-\frac{25a^2}{x^2}\right\}\div\left\{\frac{3x}{a^2}-\frac{4}{a}-\frac{5}{x}\right\}.$$

25. $$\left\{x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)\right\}\times\left\{\frac{1}{x}+\frac{1}{x^3}+\frac{1}{x^3(x^2-1)}\right\}^3.$$

26. $$\frac{\left(\frac{a+b}{a-b}\right)^2+\left(\frac{a-b}{a+b}\right)^2-2}{\left(\frac{a+b}{a-b}\right)^2+\left(\frac{a-b}{a+b}\right)^2+2}.$$

27. $$\frac{b^2}{(a+b)^2}+\frac{a^2}{(a-b)^2}-\frac{2b}{a+b}-\frac{2a}{a-b}+2.$$

28. $$\frac{x^2}{(x-y)(a+x)}+\frac{y^2-2xy}{(x-y)(a+y)}+\frac{y^2}{(a+y)^2}.$$

29. $$\frac{\frac{x}{y}+\frac{y}{x}-1}{\frac{x^3}{y^3}-1}\times\frac{\frac{x}{y}+\frac{y}{x}+1}{\frac{x^3}{y^3}+1}\times\left(\frac{x^4}{y^4}-\frac{x^2}{y^2}\right).$$

30. $$\left\{\frac{1}{x}-\frac{1}{x+\frac{1}{y+\frac{1}{z}}}\right\}\left(y+\frac{1}{x}+\frac{1}{z}\right).$$

Show that:

31. $\left(\frac{1}{a} + \frac{1}{b}\right)(a^2 + b^2 - c^2) + \left(\frac{1}{b} + \frac{1}{c}\right)(b^2 + c^2 - a^2)$
$+ \left(\frac{1}{a} + \frac{1}{c}\right)(a^2 + c^2 - b^2) = 2(a + b + c).$

32. $\frac{a + x}{x(x-y)(x-z)} + \frac{a + y}{y(y-x)(y-z)} + \frac{a + z}{z(z-x)(z-y)} = \frac{a}{xyz}.$

33. $\left(\frac{a + x}{a - x}\right)^2 - \frac{4x}{ab} = 1$, when $x = a(1 \pm \sqrt{b})$.

34. $\left(\frac{2x + a + c}{2x + b + c}\right)^2 - \frac{x + a}{x + b} = 0$, when $x = \frac{c^2 - ab}{a + b - 2c}.$

35. $\left(\frac{x + a}{x + b}\right)^3 - \frac{x + 2a + b}{x - a - 2b} = 0$, when $x = \frac{b - a}{2}.$

36. $mx + ny = p$, when $x = \frac{lp - nq}{lm - kn}$, $y = \frac{mq - kp}{lm - kn}.$

37. $(x - \overline{a + b})^2 - (x - a)(a - b) = 0$, when $x = \frac{a^2 + ab + b^2}{a + b}.$

38. $\frac{12x + 10a}{3x + a} + \frac{117a + 28x}{9a + 2x} = 18$, when $x = 3a$.

39. $\frac{a}{\sqrt{x} - \sqrt{b}} - \frac{b}{\sqrt{x} - \sqrt{a}} = \sqrt{a} - \sqrt{b}$, when $x = (\sqrt{a} + \sqrt{b})^2$.

40. $\sqrt{1 - \sqrt{1 - x}} = \frac{1 - b}{1 + b}$, when $x = 2\left(\frac{1 - b}{1 + b}\right)^2 - \left(\frac{1 - b}{1 + b}\right)^4.$

41. $\frac{a^2}{(a - b)(a - c)(a - d)} + \frac{b^2}{(b - a)(b - c)(b - d)}$
$+ \frac{c^2}{(c - a)(c - b)(c - d)} + \frac{d^2}{(d - a)(d - b)(d - c)} = 0.$

42. $\frac{x - 2y}{x + 2y} + \frac{x - 2z}{x + 2z} = 1,$

when $x = \frac{12yz}{\sqrt{(y + z)^2 + 12yz} - (y + z)}.$

CHAPTER VI.

Simple Equations.

57. When two algebraical expressions are connected by the sign (=), they are said to form an equation.

When the equality is such that it is true for *all* values of the letters in the given expressions, it is called *an identity*.

$$\text{Thus, } \left.\begin{array}{l}(x + a)(x + b) = x^2 + (a + b)x + ab \\ \text{and } (a + b)^2 - (a - b)^2 = 4ab\end{array}\right\} \text{are identities.}$$

58. When the condition of equality is such that some one or other of the letters must have particular values or a limited number of values, the statement of equality is termed an *equation of condition*, or, more briefly, *an equation*.

Thus, it may be found on trial that the equality

$$4x + 2 = 3x + 5$$

is true only when $x = 3$. Such an expression is therefore an equation.

59. The letters of an equation to which particular or a limited number of values must be given are termed *unknown* quantities.

Equations may contain *one*, *two*, *three*, or more unknown quantities.

The determination of the particular value or values of the unknown quantities is called the *solution* of the equation, and each of the values which *satisfies* the equation is said to be a *root* of the equation.

60. The expressions on the left and right sides of the sign (=) are termed the first and second sides respectively. It follows, therefore, that—

1. *If both sides of an equation be multiplied by the same quantity, the equation still subsists.*

2. *If both sides be divided by the same quantity, the equation still holds.*

3 *Any term may be transposed from one side to the other if the sign of the term be changed.*

Thus, if $3x + a = b$, we must have also

$$3x = b - a,$$

for this results from subtracting a from each side of the equation.

4. *The equation holds if every term on both sides has its sign changed.*

Thus, if $ax + b = cx - d$, we may reason as follows:—

The quantity $(ax + b)$ looked upon as a whole is given equal to the quantity $(cx - d)$ looked upon as a whole. If we change the *qualities* of these quantities, they will evidently be still equal.

$$\text{Hence, } -(ax + b) = -(cx - d)$$
$$\text{or, } -ax - b = -cx + d.$$

Now, this is the result of changing the sign of every term on both sides of the given equation.

5. *The sides of an equation may be reversed without destroying the equality.*

Thus, if $mx + n = px + q$, it must also follow that

$$px + q = mx + n.$$

6. *The sides of an equation may be raised to the* SAME POWER, *or we may extract the same root of both sides, and the equation still subsists.*

61. Simple equations are those in which the unknown quantities are not higher than the first degree, when the equations are reduced to a rational integral form.

The following is the general method adopted in solving a simple equation involving only one unknown quantity—

1. *Clear of fractions if necessary.*
2. *Transpose all the terms involving the unknown quantity to the first side of the equation, and all the remaining terms to the second side.*
3. *Simplify both sides if necessary, and divide both sides by the coefficient of the unknown quantity.*

Ex. 1. Solve the equation $5x + 6 = 3x + 12$.

Transposing the terms, we have—

$$5x - 3x = 12 - 6.$$

Now, simplifying, we get—

$$2x = 6;$$

and dividing each side by the coefficient of the unknown quantity, viz., by 2, we have—

$$x = 6 \div 2 = 3.$$

VERIFICATION.—Putting the value 3 for x in each side of the given equation, the first side becomes $5 \times 3 + 6$ or 21; and the second side becomes $3 \times 3 + 12$ or 21. The value of x found therefore *satisfies* the given equation.

Ex. 2. Given $\frac{x-2}{2} + \frac{x}{3} = 20 - \frac{x-6}{2}$, find x.

Clearing of fractions, by multiplying *every term* on each side by the L.C.M. of the denominators, viz., by 6, we get—

$$3(x-2) + 2x = 20 \times 6 - 3(x-6).$$

(Beginners often neglect to multiply integral terms such as 20 by the L.C.M.)

or $3x - 6 + 2x = 120 - 3x + 18$, or, transposing,

$$3x + 2x + 3x = 120 + 18 + 6, \text{ or, simplifying,}$$

$$8x = 144,$$

or dividing each side by 8, the coefficient of x, we have—

$$x = 18, \text{ the value required.}$$

(It will be good practice for the student to verify this result as in the last example).

Ex. 3. $\frac{4x-21}{7} + 7\frac{5}{6} + \frac{7x-28}{3} = x + 3\frac{3}{4} - \frac{9-7x}{8} + \frac{1}{12}$.

It is sometimes convenient to first *partially* clear off fractions. Thus, multiplying each side by 72, we have—

$$\frac{72(4x-21)}{7} + 47 \times 12 + 24(7x-28)$$
$$= 72x + 15 \times 18 - 9(9-7x) + 6;$$

$$\text{or } \frac{288x}{7} - 72 \times 3 + 564 + 168x - 672,$$
$$= 72x + 270 - 81 + 63x + 6;$$

or, transposing,

$$41\tfrac{1}{7}x + 168x - 72x - 63x$$
$$= 270 - 81 + 6 + 216 - 564 + 672;$$

or, simplifying,

$$74\tfrac{1}{7}x = 519; \text{ or, multiplying each side by 7,}$$
$$519x = 519 \times 7;$$
$$\therefore x = 7.$$

Ex. XV.

1. $5x + 2 = 2x + 11.$

2. $\frac{x}{4} + \frac{x}{6} + \frac{x}{8} = 2\frac{1}{6}.$

3. $2x + a = 3x - b.$

4. $3(x-7) + 4x = 2(2x - 4) + 2.$

5. $\frac{x - 1}{2} + \frac{x + 3}{4} = \frac{2x - 7}{6} + \frac{8x - 1}{12}.$

6. $\frac{7x - 8}{13} + \frac{3x - 2}{7} = \frac{10x + 3}{11} - \frac{7x - 3}{18}.$

7. $\frac{4x - 15}{9} - \frac{2x + 3}{6} = \frac{5x - 1}{12} - 3\frac{11}{18}.$

8. $\frac{8(5x + 2)}{3} - \frac{2x - 1}{8} = \frac{17x - 2}{4} + \frac{5\frac{3}{8} + 80x}{7}.$

9. $ax + bc = bx + ac.$

10. $\frac{x}{a} + \frac{a}{b} = \frac{x}{b} + \frac{b}{a}.$

11. $\frac{x - a}{b} + \frac{x - b}{a} = 2 + \frac{cx - c^2}{ab}.$

12. $abx + b^3 = b^2x + a^3.$

13. $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = ab + ac + bc.$

14. $\frac{x + b}{a} = \frac{a + x}{b}.$

15. $\frac{ax + bx + cx}{abc} = a + b + c.$

16. $x - \frac{a^2 - 3bx}{a^2} - b^2 = \frac{bx}{a} + \frac{6bx - 5a^2}{2a^2} - \frac{bx + 4a}{4a}.$

17. $\cdot 15x + \cdot 025 = \cdot 075x + \cdot 175.$

18. $\frac{3 - \cdot 125x}{8} + \cdot 1\dot{6} = \frac{2 + \cdot 1875x}{7} - \cdot 08\dot{3}.$

19. $\dfrac{1}{2x} + \dfrac{1}{12x} = \dfrac{1}{6x} + 1\tfrac{2}{3}.$

20. $(x + a)(x + b) = (x + c)(x + d).$

21. $(x - a)(x - b) = (x - \overline{a + b})^2.$

22. $\dfrac{1}{ax} + \dfrac{1}{bx} + \dfrac{1}{cx} = \tfrac{1}{2}(a + b + c)^2 - \tfrac{1}{2}\left(\dfrac{a}{bcx} + \dfrac{b}{acx} + \dfrac{c}{abx}\right).$

23. $\dfrac{x - a}{bc} + \dfrac{x - b}{ac} + \dfrac{x - c}{ab} = 2\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right).$

24. $\dfrac{1 - ax}{bc} + \dfrac{1 - bx}{ac} + \dfrac{1 - cx}{ab} = \left(\dfrac{2}{a} + \dfrac{2}{b} + \dfrac{2}{c}\right)x.$

Abbreviated Methods for Particular Cases.

62. When the unknown quantity is involved in both numerator and denominator, it is often convenient to reduce such fractions to *mixed numbers*.

Ex. Solve the equation $\dfrac{6x - 7}{x + 2} - \dfrac{12x + 18}{3x - 5} = 2.$

By division we get $\dfrac{6x - 7}{x + 2} = 6 - \dfrac{19}{x + 2};$

and $\dfrac{12x + 18}{3x - 5} = 4 + \dfrac{38}{3x - 5}.$

Hence the given equation becomes—

$$\left(6 - \frac{19}{x + 2}\right) - \left(4 + \frac{38}{3x - 5}\right) = 2;$$

$$\text{or } 6 - \frac{19}{x + 2} - 4 - \frac{38}{3x - 5} = 2;$$

$$\text{or transposing, } -\frac{19}{x + 2} = \frac{38}{3x - 5} + 2 - 6 + 4;$$

$$\text{or } -\frac{19}{x + 2} = \frac{38}{3x - 5};$$

or, dividing each side by -19, we have—

$$\frac{1}{x + 2} = -\frac{2}{3x - 5}.$$

Hence, multiplying each side by the L.C.M. of the denominators—

$$3x - 5 = -2(x + 2) = -2x - 4,$$
$$\text{or } 3x + 2x = -4 + 5,$$
$$\text{or } 5x = 1,$$
$$\therefore x = \tfrac{1}{5}.$$

63. *When each side of an equation consists solely of a single fraction, the numerator of either fraction may change places with the denominator of the other.*

Let $\frac{a}{b} = \frac{p}{q}$ be the equation.

Multiply each side by b, then, by Art. 60 (1.)—

$$\frac{a}{b} \times b = \frac{p}{q} \times b, \text{ or } a = \frac{pb}{q}.$$

Divide each side by p, then, by Art. 60 (2.)—

$$\frac{a}{p} = \frac{pb}{q} \div p, \text{ or } \frac{a}{p} = \frac{b}{q}.$$

Here the denominator b of the first side of the given equation has changed places with the numerator p of the second side.

And similarly we may show that $\frac{q}{b} = \frac{p}{a}$, where the other numerator and denominator have changed places.

Cor. *The two sides of an equation of the form* $\frac{a}{b} = \frac{p}{q}$ *may be inverted.*

For interchanging p and b in the last result, viz., $\frac{q}{b} = \frac{p}{a}$, we get $\frac{q}{p} = \frac{b}{a}$, and therefore also, by Art. 60 (5.), we have $\frac{b}{a} = \frac{q}{p}$.

(The student is *cautioned* against inverting the *separate terms* of the two sides of an equation when there are more than one term on each side.)

64. When each side of an equation consists solely of a *single* fraction, we may perform the following operations:—

1. *We may add or subtract the numerator and denominator of* EACH *fraction for a new numerator or denominator, and retain either the original numerator or denominator for the other term of the fraction, both sides being always similarly treated.*

Thus, if $\frac{a}{b} = \frac{p}{q}$, we have—

$$\text{(i.) } \frac{a+b}{b} = \frac{p+q}{q}, \quad \text{(ii.) } \frac{a-b}{b} = \frac{p-q}{q},$$

$$\text{(iii.) } \frac{a+b}{a} = \frac{p+q}{p}, \quad \text{(iv.) } \frac{a-b}{a} = \frac{p-q}{p},$$

or, (v.) we may have equations formed by inverting each of these.

These results are easily obtained—

For, since $\frac{a}{b} = \frac{p}{q}$, we have, adding unity to each side—

$$\frac{a}{b} + 1 = \frac{p}{q} + 1 \text{ or } \frac{a+b}{b} = \frac{p+q}{q}.$$

And so, by subtracting unity from each side, we get—

$$\frac{a-b}{b} = \frac{p-q}{q}, \text{ and so on.}$$

2. *We may take the* SUMS *of the numerator and denominator of each for new numerators or denominators, and the* DIFFERENCES *for the other terms of the fraction; and* VICE VERSA, *both sides being always similarly treated.*

Thus, if $\frac{a}{b} = \frac{c}{d}$, we have also $\frac{a+b}{a-b} = \frac{c+d}{c-d}$,

$$\text{and } \frac{a-b}{a+b} = \frac{c-d}{c+d},$$

for we have just shown that $\frac{a+b}{b} = \frac{p+q}{q}$,

$$\text{and } \frac{a-b}{b} = \frac{p-q}{q}.$$

Hence, dividing equals by equals, we get—

$$\frac{a+b}{b} \div \frac{a-b}{b} = \frac{p+q}{q} \div \frac{p-q}{q},$$

or $\dfrac{a+b}{a-b} = \dfrac{p+q}{p-q}$, which is the first result.

And inverting each side, we have, by Art. 63 (Cor.)—

$$\frac{a-b}{a+b} = \frac{p-q}{p+q}.$$

Ex. 1. Solve the equation $\dfrac{mx+a+b}{nx-c-d} = \dfrac{mx+a+c}{nx-b-d}$.

By Art. 63, we have $\dfrac{mx+a+b}{mx+a+c} = \dfrac{nx-c-d}{nx-b-d}$.

Then, by Art. 64 (1.), retaining the numerators and taking the *differences* for new denominators, we have—

$$\frac{mx+a+b}{b-c} = \frac{nx-c-d}{b-c};$$

or, multiplying each side by $(b-c)$—

$mx + a + b = nx - c - d$; or, transposing—

$$(m-n)\,x = -\,(a+b+c+d); \quad \therefore x = -\,\frac{a+b+c+d}{m-n}.$$

Ex. 2. Solve $\dfrac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = a.$

We may consider the quantity a as a fraction whose denominator is *unity*, or as $\dfrac{a}{1}$.

Then, Art. 64 (2.), taking the *sum* and *difference*, we have—

$$\frac{2\sqrt{a+x}}{2\sqrt{a-x}} = \frac{a+1}{a-1}; \text{ or—}$$

$$\frac{\sqrt{a+x}}{\sqrt{a-x}} = \frac{a+1}{a-1}; \text{ or, squaring—}$$

$$\frac{a+x}{a-x} = \frac{a^2+2a+1}{a^2-2a+1}.$$

Again, taking the *difference* and *sum*, we have—

$$\frac{2x}{2a} = \frac{4a}{2a^2 + 2}; \text{ or—}$$

$$\frac{x}{a} = \frac{2a}{a^2 + 1}.$$

$$\therefore x = \frac{2a^2}{a^2 + 1}.$$

65. We now give an example to show that sometimes the easy solution depends on an advantageous arrangement of the terms on the two sides of the equation.

Ex. Solve $\sqrt{x + 4} + \sqrt{x - 3} = 7$.

Transposing, we have—

$\sqrt{x + 4} = 7 - \sqrt{x - 3}$; squaring, then—

$x + 4 = 49 - 14\sqrt{x - 3} + (x - 3)$;

subtracting x from each side and transposing, then—

$14\sqrt{x - 3} = 49 - 3 - 4 = 42$.

$\therefore \sqrt{x - 3} = 3$; or squaring,

$x - 3 = 9$; $\therefore x = 3 + 9 = 12$.

Should the student commence by squaring at once, he will render the equation more complicated.

Ex. XVI.

1. $\frac{3x + 7}{x + 4} = \frac{3x - 13}{x - 4}$.

2. $(x - a)(x - b) = (x - c)(x - d)$.

3. $\frac{3x + 13}{15} - \frac{3x + 10}{5x - 50} = \frac{x}{5}$.

4. $\frac{1 - 25x}{15} - \frac{3 - 2\frac{1}{2}x}{14(x - 1)} = \frac{28 - 5x}{3} - \frac{10x - 11}{30} + \frac{x}{3}$.

5. $\frac{x - 4}{6x + 5} + \frac{3x - 13}{18x - 6} = \frac{1}{3}$.

6. $\frac{3 - 2x}{1 - 2x} - \frac{5 - 2x}{7 - 2x} = 1 - \frac{4x^2 - 2}{7 - 16x + 4x^2}$.

7. $\dfrac{18x - 22}{13 - 2x} + 6x + \dfrac{1 + 16x}{8} = 13\tfrac{1}{4} - \dfrac{101 - 64x}{8}.$

8. $\dfrac{6x + 5}{3x + 1} + \dfrac{58\frac{1}{2} + 14x}{9 + 2x} = 9.$

9. $\dfrac{3}{4x - 9} + \dfrac{1\frac{1}{2}}{6x - 21} = \dfrac{1}{x - 2\frac{1}{2}}.$

10. $\dfrac{4x - 7}{2x - 9} + \dfrac{2 - 14x}{7} + \dfrac{3\frac{1}{3} + x}{14} = \dfrac{10 - 3\frac{6}{7}x}{2} - \dfrac{19}{21}.$

11. $\dfrac{6x^2 - 7x - 6\frac{2}{3}}{2x - 3} - \dfrac{9x^2 - 12x - 19}{3x - 5} = \dfrac{17}{6x - 7}.$

12. $\dfrac{ax + m + 1}{ax + m - 1} + \dfrac{ax + n}{ax + n - 2} = \dfrac{ax + m}{ax + m - 2} +$
$\dfrac{ax + n + 1}{ax + n - 1}.$

13. $\dfrac{\sqrt{x - a + b} - \sqrt{x + a - b}}{\sqrt{x - a + b} + \sqrt{x + a - b}} = \dfrac{a - b.}{a + b}$

14. $\dfrac{1 - \sqrt{1 - \sqrt{1 - x}}}{1 + \sqrt{1 - \sqrt{1 - x}}} = b.$

15. $\sqrt{2x + 10} + \sqrt{2x - 2} = 6.$

16. $\sqrt{8 - x} - \dfrac{3}{\sqrt{1 - x}} = \sqrt{1 - x}.$

17. $\sqrt[3]{1 + \sqrt{x}} + \sqrt[3]{1 - \sqrt{x}} = 2.$

18. $\dfrac{ax + 1 + \sqrt{a^2x^2 - 1}}{ax + 1 - \sqrt{a^2x^2 - 1}} = b.$

19. $\dfrac{a}{\sqrt{x} - \sqrt{b}} - \dfrac{b}{\sqrt{x} - \sqrt{a}} = \dfrac{a - b}{\sqrt{x}}.$

20. $\dfrac{a}{\sqrt{x} + \sqrt{a}} = \dfrac{a}{\sqrt{x} - \sqrt{a}} + \sqrt{a}.$

21. $\sqrt{\dfrac{\sqrt{x} + a}{\sqrt{x} - a}} - \sqrt{\dfrac{\sqrt{x} - a}{\sqrt{x} + a}} = \sqrt{x - a^2}.$

22. $\dfrac{1}{x-2} - \dfrac{2 + \frac{5}{2}x^2 - \frac{1}{2}x^3}{6 - 5x + x^2} - \frac{1}{2}x = \dfrac{5}{x-3}.$

23. $\dfrac{4x+5}{x+1} + \dfrac{x+5}{x+4} = \dfrac{2x+5}{x+2} - \dfrac{x^2-10}{x+3} + x.$

24. $\dfrac{ax+b}{cx+d} + \dfrac{cx+e}{ax+f} = \dfrac{a^2+c^2}{ac}.$

25. If $a + \sqrt{a^2+x^2} = b + x$, then $x = \dfrac{b}{2}\left(\dfrac{a}{b-a} - 1\right).$

26. If $\sqrt{x} - \sqrt{a - \sqrt{ax + x^2}} = \frac{1}{2}\sqrt{a}$, then $x = \frac{9}{16}a.$

27. If $\dfrac{1+x}{1-x} = \dfrac{b}{a} : \dfrac{1+x+x^2}{1-x+x^2}$, then $x = \sqrt[3]{\dfrac{b-a}{b+a}}.$

28. If $\dfrac{a + c(a-x)}{a + c(a+x)} = \dfrac{a}{a+2cx} + \dfrac{a-x}{x}$, then $x = a.$

29. If $\dfrac{a+x}{\sqrt{2ax+x^2}} = \dfrac{b+1}{b-1}$, then $x = \dfrac{a}{2\sqrt{b}}(\sqrt{b}-1)^2.$

30. If $\dfrac{a(1+b)+bx}{a(1+b)-bx} = \dfrac{a}{a-2cx} - \left(\dfrac{a}{x}+1\right)$, then $x = -a.$

31. If $\dfrac{x^2-3x-9}{x-5} + \dfrac{x^2-7x-17}{x-9}$

$= \dfrac{x^2-6x-15}{x-8} + \dfrac{x^2-4x-11}{x-6}$, then $x = 7.$

32. If $\sqrt{x+m} + \sqrt{x+n} = \sqrt{m} + \sqrt{n}$, then $x = 0.$

33. If $\dfrac{x^2-17}{x-4} + \dfrac{2x^2+7x-13}{2x-3}$

$= \dfrac{2x^2+x-30}{2x-7} + \dfrac{x^2+4x-4}{x-1}$, then $x = \frac{5}{2}.$

34. If $\left(\dfrac{x+\cdot 5}{x+\cdot 7}\right)^{\frac{1}{2}} + \dfrac{\cdot 1}{x+\cdot 6} = 1$, then $x = -\frac{1}{2}.$

35. If $\sqrt{x+3{\cdot}2} - \sqrt{x+\cdot 8} = 2\sqrt{x-1{\cdot}6}$, then $x = 1{\cdot}7.$

36. If $\dfrac{2{\cdot}7\,x - 1}{\sqrt{2{\cdot}7\,x} + 1} = \dfrac{\sqrt{2{\cdot}7\,x} + 7}{2}$, then $x = 30$.

37. If $\sqrt{4\,x + {\cdot}01} + \sqrt{4\,x - {\cdot}36} = 3{\cdot}7$, then $x = {\cdot}9$.

38. If $\left(\dfrac{m - x}{m + x}\right)^2 = 1 - \dfrac{nx}{mp}$, then $x = m\left(\pm\, 2\sqrt{\dfrac{p}{n}} - 1\right)$.

Problems producing Simple Equations involving One Unknown Quantity.

66. To solve an algebraical problem we represent the required or unknown quantity by a letter, as x, and then express the given conditions in algebraical language. Thus we form an equation, the solution of which gives the required value of the unknown quantity.

Ex. 1. My purse and money are together worth 24 shillings, and the money is worth seven times the purse. Find the value of each.

Let x = the value in shillings of the purse,
Then $7\,x$ = „ „ money.

Now, by problem the value of both together is 24 shillings. Hence we have—

$$x + 7\,x = 24$$
$$\text{or } 8\,x = 24$$
$$\therefore\ x = 3, \text{ the value in shillings of the purse,}$$
$$\text{and } \therefore \text{ also } 7\,x = 7 \times 3 = 21, \quad „ \quad \text{money.}$$

Ex. 2. What number is that to which, if 36 be added, the sum shall be equal to 3 times the number?

Let x = the number;
$\therefore\ x + 36$ = the sum when 36 is added,
and $3\,x$ = 3 times the number.

Hence, by problem—

$$x + 36 = 3\,x,$$
$$\text{or } x - 3\,x = -\,36,$$
$$\text{or } \quad -\,2\,x = -\,36;$$
$$\therefore x = \frac{-\,36}{-\,2} = 18, \text{ the number required.}$$

Ex. 3. The distance between two towns is such that a train, whose speed is 30 miles an hour, takes 1 hour more in going 10 miles over 5 times the distance than a train whose speed is 20 miles an hour takes in going within 4 miles of 3 times the distance Find the distance between the towns.

Let x = the distance required in miles.

Then $5x + 10 = 5$ times the distance together with 10 miles, and $\frac{5x + 10}{30}$ = time in hours to travel this distance at 30 miles an hour.

And so, $\frac{3x - 4}{20}$ = time in hours to travel 4 miles less than 3 times the required distance, at 20 miles an hour.

But by the problem the former of these times exceed the latter by 1 hour.

Hence $\frac{5x + 10}{30} - \frac{3x - 4}{20} = 1.$

From this we easily find $x = 28$.

Hence 28 miles is the distance required.

Ex. 4. Find the price of an article, when as many can be bought for 1s. 4d. as can be bought for 2s. after the price has been raised 1d.

Let x = the price required in *pence;*

Then $\frac{16}{x}$ = number of articles bought for 1s. 4d.

And $x + 1$ = the raised price in *pence.*

$\therefore \frac{24}{x + 1}$ = number of articles bought for 2s. at the raised price.

But, by the problem, the number of articles in each case is the same.

Hence $\frac{16}{x} = \frac{24}{x + 1}$, from which $x = 2$.

Hence 2d. is the price required.

Ex. 5. A man sells geese at as many shillings each as the number he has, and having returned 5s., finds that if he had

had 2 more to sell on the same condition, and had returned 3s., he would have had 38s. more. How many had he?

Let x = the number required;

Then $x^2 - 5$ = number of shillings received.

Also, on the second supposition,

$(x + 2)^2 - 3$ = number of shillings he would have received.

Now, by the problem, this latter number is 38 more than the former.

Hence $(x + 2)^2 - 3 = x^2 - 5 + 38$, from which we find $x = 8$.

Ex. 6. A waterman finds that he can row 5 miles in $\frac{3}{4}$ hour with the tide, and that it takes him $1\frac{1}{2}$ hours to row the same distance against the tide when it is but half as strong. What is the velocity of the tide?

Let x = the velocity of the tide in miles per hour.

Now the velocity of the boat when going *with* the tide $= 5 \div \frac{3}{4} = \frac{20}{3}$.

$\therefore \frac{20}{3} - x$ = velocity of the boat when there is no tide.

Again, velocity of boat *against* the tide when it is half as strong $= 5 \div 1\frac{1}{2} = \frac{10}{3}$.

$\therefore \dfrac{10}{3} + \dfrac{x}{2}$ = velocity of the boat, when there is no tide.

Hence we have—

$$\frac{10}{3} + \frac{x}{2} = \frac{20}{3} - x; \text{ from which}$$
$$x = 2\tfrac{2}{9}.$$

$\therefore$ The velocity required is $2\frac{2}{9}$ miles per hour.

Ex. XVII.

1. If I add 25 to 3 times a certain number, I obtain the same result as if I subtract 25 from 8 times the number. Find the number.

2. Divide 70 into 2 such parts that the one shall be as much above half the number as the other is above 15.

3. Divide £720 among A, B, and C, so that B may have twice as much as C, and A as much as B and C together.

4. There are two trains, one of which goes 5 miles an hour faster than the other, and the former performs a journey of 100 miles, while the latter goes 75 miles. Find their respective rates.

5. A horse when let out for hire brings in *a clear gain* of 10s. per day, but costs 1s. 6d. daily for food. At the end of 30 days his master had gained £11. 11s. Required the number of days for which he was hired.

6. A and B have 4 guineas between them, and play at hazard. B loses $\frac{1}{6}$ of his money, and afterwards gains $\frac{1}{10}$ of what he then had. It is then seen that B has as much money as A had at the end of the first game. How much had each at first?

7. A and B have respectively an equal number of florins and crowns. B pays a debt of 4s. to A, and then A's money is just half B's. Find what each had.

8. A workman, instead of adopting the 9 hours' system, worked 10 hours daily, and had a corresponding rise of wages. By this means his wages were increased 4s. weekly. Find his original wages.

9. A person who has regular wages of 26s. weekly, thinking to better himself, takes a job at higher wages. He is, however, put on half-time during 20 weeks of the year, and finds himself at the end of the year £4. 12s. worse off. Required his increased wages.

10. A company of men, arranged in a hollow square 4 deep, numbered 144. What was the number in a side of the square?

11. In an examination paper there were two series of questions, and the questions of the second series carried each 3 marks more than those of the first series. A candidate who attempted 3 of the first series, obtaining half marks for them, and 5 of the second series, obtaining for these full marks, got altogether 80 marks. Find the number of marks attached to each question of the first series.

12. A grocer has tea at 3s. 4d. and at 4s. He sells altogether 64 pounds, thereby realizing £12. How much did he sell of each?

13. If 11 be subtracted from 5 times a certain number, and the remainder divided by 6, the quotient will exceed by 2 the quotient obtained by subtracting 3 from 4 times the number and dividing the remainder by 7. Find the number.

14. A garrison of 1,250 men were provisioned for 64 days; but after 22 days a certain number were called away, and it was found that the remaining provisions lasted the number left for 70 days. Find the number told off.

15. At a railway station £15 was taken for single fares, and £33. 15s. for returns. The number of return tickets exceeded the single tickets by 10, and the price of a return ticket was half as much again as a single ticket. Find the fare for a single journey.

16. In a tour lately made round the world, the distance travelled by water was 20,000 miles, and by land 8,000 miles; and the whole time taken was 220 days. Supposing the rate by water to be two-thirds of that by land, find the number of days travelled by land.

17. The distance between A and B is 32 miles. A person starting from A, at the rate of 4 miles an hour, meets another who started from B half an hour later, at a rate of 3$\frac{1}{2}$ miles an hour. At what point will they meet?

18. There are two clocks, one of which gains twice as much per day as the second loses, and they are set right at noon on Monday. When it is noon on Thursday by the first clock, it is 11·50 A.M. by the second. What is the gain per day of the first clock?

19. A draper raises his goods a certain rate per cent., and afterwards reduces them to the original price by lowering them 13$\frac{1}{23}$ per cent. Find the original rise per cent.

20. Required the distance between two towns such that a person can perform the journey one hour sooner when he walks 4 miles an hour than when he walks 3$\frac{1}{2}$ miles an hour?

21. The sum of £12. 15s. is paid away with an equal number of sovereigns, crowns, and sixpences. Required the number of each.

22. A walks along an inclined plane at a certain rate, and B walks along the base of the plane at a rate of one-third of

a mile per hour less than A. The inclination of the plane is such that A is always vertically over B, and that at the end of half an hour they are exactly five-sixths of a mile apart. Find the respective rates of A and B.

23. There is a direct road over a hill between two stations at the foot of each side. The distance on the one side from the foot to the top is 5 miles, and the road down the other side forms a right angle with the road up. It is also known to be 1 mile less down the hill than the direct distance by tunnel between the two stations. Find the distance down the hill.

24. Two trains, whose respective lengths are 122 and 98 yards, and the former of which is going at the rate of 35 miles an hour, pass each other in 30 seconds. Find the rate and relative direction of the second train.

25. A man bought a number of sheep for £132, and having lost 10, and sold 20 that were diseased at 6s. per head below cost price, disposed of the remainder for £116, thereby realizing his outlay. How many did he buy?

26. A boy spends 10s. in oranges and apples. The oranges were bought at 5 for 6d., and the apples at 3 for 2d.; and their number together amounted to 132. What did he spend on each?

27. If B is allowed 2 hours more time than A takes to do a piece of work, he will do 4 times as much, and if C is allowed 1 hour more than A, he also can do 4 times as much. Moreover, D requires 4 hours more than A to do the piece of work. Also, the work done by A and B together is the same as that done by C and D together in the same time. Required the respective times for A, B, C, D to do a piece of work.

28. A person sells out £1,200 Three and a Half per Cent. stock, and invests the money in Two and a Half per Cents., whose price is 14 lower than the first-named stock. The loss in annual income is £7. Find the price of the first-named stock.

29. The banker's discount on a certain sum of money at 5 per cent. per annum is equal to the true discount on a sum £50 larger. Find the sum.

30. An express train, which ought to perform its journey in $2\frac{1}{2}$ hours, after having gone uniformly 80 miles, finds itself 6 minutes behind. However, by increasing the speed to as many miles per hour as there were miles in half the journey, it just arrived at its destination in time. Find the original speed of the train, and the length of the journey.

31. A vessel contains a quantity of spirit (sp. gr. ·9) and water, and a cylinder of wood (sp. gr. ·92), whose length is 10 inches, floats upright, so as to be just covered by the spirit. Find how much of the cylinder floats in the water.

32. A mixture of hydrogen and oxygen is found to condense when fired, to 16 vols. of steam. Now, every 3 vols. of such a mixture is known to condense to 2 vols., when the original gases are in the proportion of 2 : 1. Find the quantity of each.

33. A mixture of 100 grams of sodic and potassic sulphates yielded a gram of baric sulphate. Now, each gram of sodic sulphate yields b grams of baric sulphate, and each gram of potassic sulphate yields c grams of baric sulphate. Find the amount of sodic and potassic sulphates in the mixture.

34. If a oxen consume b acres of grass in c weeks, and a' oxen consume b' acres of grass in c' weeks, the grass growing uniformly, find the week's growth of an acre.

35. The freezing and boiling points of a common thermometer are marked 32° and 212° respectively; those on the centigrade thermometer are marked 0° and 100°. At what temperature do the graduations agree?

36. A person going at the rate of a miles per hour finds himself b hours too late when he has c miles farther to go. How much must he increase his speed to reach home in time?

Simultaneous Equations of the First Degree with two Unknown Quantities.

67. Suppose we have given the equation $3x - 4y = 5$, then by ascribing to y a series of values we get a corresponding series of values for x.

Thus we may have $\left.\begin{matrix} x = 3 \\ y = 1 \end{matrix}\right\}, \left.\begin{matrix} x = 7 \\ y = 4 \end{matrix}\right\}, \left.\begin{matrix} x = 11 \\ y = 7 \end{matrix}\right\}$, &c.

Again, if another equation, as $4x + y = 32$, be given, we may in the same way obtain a series of pairs of values which satisfy it. And, further, if the two equations are *distinct* and *compatible*, there is always *a pair of values common to the two equations.* This pair of values then satisfies both equations, and the equations are called **simultaneous** equations.

The methods of solving simultaneous equations will now be explained.

First Method.—*Equalize the coefficients of one of the unknown quantities in both equations, and add or subtract the equations so obtained, so as to obtain an equation with one unknown quantity.*

Ex. $$\left.\begin{array}{l} 4x + 3y = 17 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1) \\ 9x - 5y = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2) \end{array}\right\}.$$

From (1), multiplying each side by 9, we get—

$$36x + 27y = 153 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

And from (2), multiplying each side by 4, we get—

$$36x - 20y = 12 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

(3) − (4), then $47y = 141$

$$\therefore y = 3.$$

Hence, substituting in (1), we have—

$4x + 3 \times 3 = 17$, from which we get—

$$x = 2.$$

Hence, the solution required is $x = 2$, $y = 3$.

Second Method.—*Express one of the unknown quantities in terms of the other by means of either equation, and substitute its value in the other.*

Taking the same example, we have—

$$\left.\begin{array}{l} 4x + 3y = 17 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1) \\ 9x - 5y = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2) \end{array}\right\}.$$

From (1) we have $4x = 17 - 3y$, or $x = \dfrac{17 - 3y}{4} \ldots (3)$.

Substituting this value of x in (2) we have—

$$9\left(\frac{17 - 3y}{4}\right) - 5y = 3,$$

or, $153 - 27y - 20y = 12$, from which—

$$y = 3.$$

Then from (3) we get $x = \dfrac{17 - 3 \times 3}{4} = 2.$

THIRD METHOD.—*Express one of the unknown quantities in terms of the other by means of each equation, and equate the expression.*

We will this time express in each case y in terms of x.

$$\text{We have } \left.\begin{array}{r} 4x + 3y = 17 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1) \\ 9x - 5y = 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2) \end{array}\right\}.$$

From (1) $3y = 17 - 4x$, or $y = \dfrac{17 - 4x}{3} \ldots(3).$

And from (2) $5y = 9x - 3$, or $y = \dfrac{9x - 3}{5} \ldots\ldots(4).$

Equating (3) and (4), then $\dfrac{17 - 4x}{3} = \dfrac{9x - 3}{5}$, from which

$$x = 2.$$

Then from (3) by substitution, $y = \dfrac{17 - 4 \times 2}{3} = 3.$

Simultaneous Equations of the First Degree of Three or more Unknown Quantities.

68. In the case of *three* unknown quantities we may obtain, from the three given equations, two equations with two unknown quantities, and then, by a similar method, from the two obtain an equation with one unknown quantity; and a like method may be pursued for more than three unknown quantities.

$$\begin{array}{lrcll}
\text{Ex.} & 3x + y + 4z & = & 25 \ldots\ldots\ldots\ldots & (1) \\
 & 4x + 3y - 5z & = & -3 \ldots\ldots\ldots\ldots & (2) \\
 & 6x + 7y - 8z & = & 1 \ldots\ldots\ldots\ldots & (3) \\
\text{From (1)} & 12x + 4y + 16z & = & 100 \ldots\ldots\ldots\ldots & (4) \\
\text{And from (2)} & 12x + 9y - 15z & = & -9 \ldots\ldots\ldots\ldots & (5) \\
(5)-(4)\text{, then} & 5y - 31z & = & -109 \ldots\ldots\ldots\ldots & (6) \\
\text{Again, from (1)} & 6x + 2y + 8z & = & 50 \ldots\ldots\ldots\ldots & (7) \\
(3)-(7)\text{, then} & 5y - 16z & = & -49 \ldots\ldots\ldots\ldots & (8) \\
(8)-(6)\text{, then} & 15z & = & 60 & \\
 & \therefore z & = & 4 &
\end{array}$$

Hence from (8), by substitution—

$$5y - 16 \times 4 = -49,$$
$$\text{from which } y = 3.$$

And hence from (1), by substitution of the known values of y and z—

$$3x + 3 + 4 \times 4 = 25,$$
$$\text{from which } x = 2.$$

Hence the respective values of x, y, z, are 2, 3, 4.

Ex. XVIII.

1. $6x + y = 22$, $5x + 3y = 27$.
2. $4x - 3y = 14$, $6x + 5y = 40$.
3. $3x + 5y = 44$, $y - x = 4$.
4. $\frac{x}{4} + \frac{y}{3} = 2\frac{2}{3}$, $\frac{3x}{5} + \frac{y}{10} = 2\frac{9}{10}$.
5. $\frac{x}{6} + \frac{y}{4} = 4$, $\frac{x}{4} + \frac{y}{6} = 4\frac{1}{3}$.
6. $\frac{x}{8} + \frac{y}{6} = 1\frac{1}{4}$, $x + \frac{y}{3} = 4$.
7. $3x + 4y + z = 14$, $2x + y + 5z = 19$, $5x + 2y + 3z = 18$.
8. $7x + 2y + 3z = 20$, $3x - 4y + 2z = 1$, $5y - 2x + 7z = 29$.
9. $2x - 5y = 3$, $3y - 2z = -1$, $4x + 2z = 20$.
10. $ax + by = c$, $a_1x + b_1y = c_1$.
11. $x + y = a$, $y + z = b$, $x + z = c$.
12. $ax + by = d$, $by + cz = e$, $ax + cz = f$.
13. $\frac{3x - 2y}{4} + \frac{5x + 2y}{6} = 5\frac{1}{2}$, $\frac{x + 2y}{7} - \frac{3x - 2y}{2} = 1$.
14. $\frac{x + y}{10} + \frac{x - y}{15} = \frac{23}{30}$, $\frac{x + y}{15} + \frac{x - y}{10} = \frac{7}{30}$.
15. $\frac{3}{x} + \frac{9}{y} = 1\frac{7}{8}$, $\frac{15}{x} - \frac{11}{y} = 2\frac{3}{8}$.

16. $\dfrac{4}{x-2} + \dfrac{6}{y} = 5\tfrac{1}{2}$, $\dfrac{4}{y} - \dfrac{1}{x-2} = 0$.

17. $4(x + 3y) - 3(x + y) = \tfrac{1}{3}(x + y)(x + 3y)$, $\dfrac{20}{x+y} + \dfrac{9}{x+3y} = 3$.

18. $73y - 5x = (x - 5y)(x + 3y)$, $\dfrac{2}{x-5y} - \dfrac{5}{x+3y} = \dfrac{7}{33}$.

19. $\dfrac{1}{x} + \dfrac{1}{y} = a$, $\dfrac{1}{y} + \dfrac{1}{z} = b$, $\dfrac{1}{x} + \dfrac{1}{z} = c$.

20. $3x - 2y = 0$, $4y - 3z = 0$, $5z - 6u = 14$, $7u - 2x = 3$.

21. $3x - 2y = 5z - 6y = 7x - 4z = 1$.

22. $x + y + z = a$, $y + z + u = b$, $z + u + x = c$, $u + x + y = d$.

23. $\dfrac{x}{m} + \dfrac{y}{n} = a$, $\dfrac{y}{n} + \dfrac{z}{p} = b$, $\dfrac{x}{m} + \dfrac{z}{p} = c$.

24. $x + ay + bz = \alpha$, $y + az + bx = \beta$, $z + ax + by = \gamma$.

25.
$$\begin{aligned} x + ay + a^2z + a^3 &= 0. \\ x + by + b^2z + b^3 &= 0. \\ x + cy + c^2z + c^3 &= 0. \end{aligned}$$

26.
$$\begin{aligned} x + ay + a^2z + a^3u + a^4 &= 0. \\ x + by + b^2z + b^3u + b^4 &= 0. \\ x + cy + c^2z + c^3u + c^4 &= 0. \\ x + dy + d^2z + d^3u + d^4 &= 0. \end{aligned}$$

27.
$$\begin{aligned} x + ay &= \alpha. \\ y + bz &= \beta. \\ z + ct &= \gamma. \\ t + dx &= \delta. \\ u + ex &= \varepsilon. \end{aligned}$$

28.
$$\begin{aligned} \frac{a}{x} + \frac{b}{y} + \frac{c}{z} &= d \\ \frac{a'}{x} + \frac{b'}{y} + \frac{c'}{z} &= d' \\ \frac{a''}{x} + \frac{b''}{y} + \frac{c''}{z} &= d'' \end{aligned}$$

29.
$$\begin{aligned} x_1 + 2x_2 + 3x_3 + \text{\&c.} + nx_n &= a_1. \\ x_2 + 2x_3 + 3x_4 + \text{\&c.} + nx_1 &= a_2. \\ \text{\&c.} &= \text{\&c.} \\ x_n + 2x_1 + 3x_2 + \text{\&c.} + nx_{n-1} &= a_n. \end{aligned}$$

30. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{x}{b} + \frac{y}{c} + \frac{z}{a} = \frac{x}{c} + \frac{y}{a} + \frac{z}{b} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$

Problems producing Simultaneous Equations of the First Degree.

69. There are certain problems which may be solved with much greater facility by the introduction of more than one unknown quantity.

Ex. 1. Nine men and seven women receive together £3. 2s. 8d., and three men receive 10d. more than four women. Find the receipts of each.

Let x and y respectively represent in pence the receipts of a man and woman.

Then, since £3. 2s. 8d. = 752d., expressing the conditions of the problem in algebraical language—

$$\left.\begin{aligned} 9x + 7y &= 752 \\ 3x - 4y &= 10 \end{aligned}\right\} \text{Solving these equations, we find—}$$

$$x = 54,\ y = 38.$$

Hence, the receipts of a man and woman are respectively 54d. and 38d., or 4s. 6d. and 3s. 2d.

Ex. 2. Find a fraction such that, if we diminish its numerator by 1, it becomes equal to $\frac{1}{7}$; and if we increase its denominator by 1, it becomes equal to $\frac{1}{6}$.

Let $\frac{x}{y}$ be the required fraction.

$$\text{Then, by problem, } \left.\begin{aligned} \frac{x-1}{y} &= \frac{1}{7} \\ \text{and } \frac{x}{y+1} &= \frac{1}{6} \end{aligned}\right\}, \text{ which equations, when solved, give—}$$

$$x = 6,\ y = 35.$$

$\therefore \frac{6}{35}$ is the fraction required.

Ex. 3. A's money, together with twice B's and thrice C's, amounts to £38; B's money, together with twice C's and thrice A's, to £35; and C's money, together with twice A's and thrice B's, also to £35. Find the money of each.

Let x, y, z be respectively the number of pounds each has. Then we have—

$$\left.\begin{aligned} x + 2y + 3z &= 38 \\ y + 2z + 3x &= 35 \\ z + 2x + 3y &= 35 \end{aligned}\right\}; \text{ from which we get—}$$

$$x = 5, y = 6, z = 7.$$

Hence, £5, £6, £7 are the respective moneys of A, B, and C.

Ex. XIX.

1. A and B engage in play. A puts down half-a-crown to B's florin. They play twenty games, and then it is found that A has won 4s. How many games did each win?

2. There is a number, the sum of whose two digits is 10, and, if 36 be added to the number, the digits change places. Find the number.

3. A grocer has tea at 3s. 4d. a lb., and sugar at 4s. a stone. He sells £2 worth of the two. If he had raised the tea 20 per cent., and lowered the sugar 12½ per cent., and sold the same quantities of each, his profits would have been 1s. 6d. more. Find the quantity sold of each.

4. Eight times the numerator of a certain fraction exceeds three times the denominator by 3, and five times the numerator added to twice the denominator make 29. Find the fraction.

5 If the number of cows in a field were doubled, there would be 65 cows and horses together; but, if the number of horses be doubled, and that of the cows halved, there would be 46. How many are there of each?

6. Thirty shillings are spent in brandy, and 42s. in gin; 19 bottles being purchased in all. Had the 42s. been spent in brandy, and the 30s. in gin, 17 bottles only would have been bought. Find the cost per bottle of each.

7. A fishmonger receives 240 mackerel. He sells a certain number at 4 for a shilling, but the rest being seized as bad fish, and he being fined 10s., finds himself a loser by 9s. Had he sold them at 3 for a shilling, he would have been a gainer by 5s., if 13 more fish had been seized. How many did he sell, and what did he pay for the lot?

8. Three persons invest their money at 3, 4, 5 per cent. interest respectively. The total amount of interest is £38, and the interest of the first and third together is $2\frac{1}{8}$ that of the second; while the total interest would have been £34 had the rates been 5, 4, 3 per cent. respectively. Find the capital of each.

9. A toll-gate keeper receives 8s. 8d. for the toll of a number of horses, oxen, and sheep, the tolls for each being respectively $1\frac{1}{2}$d. 1d., $\frac{1}{2}$d. Had there been twice as many sheep and the number of horses diminished accordingly, he would have received 7s. 2d. Had the oxen passed through free, and the tolls for a horse and sheep respectively been 2d. and $\frac{3}{4}$d., he would have received 9s. $1\frac{1}{2}$d. Find the number of each.

10. A, B, C start from the same place. B after a quarter of an hour doubles his rate, while C, who falls, after ten minutes diminishes his rate $\frac{1}{6}$th. At the end of half an hour A is $\frac{1}{4}$ mile before B, and $\frac{1}{2}$ mile before C, and it is observed that the total distance which would have been walked by the three, had they each continued to walk uniformly from the first, is $6\frac{1}{4}$ miles. Find the original rate of each.

11. A, B, C, working 3, 4, 5 hours respectively, can do $2\frac{1}{24}$ pieces of work; if they each work an hour more, they can finish an extra $\frac{13}{24}$ of a piece; and, if C does not work, the other two, working for 1 and 6 hours respectively, can together finish 1 piece. Find the time required for A, B, C to finish separately a piece of work.

12. There are three numbers such that, if the first be increased by 6, and the second diminished by 5, the product of the results is the product of the first two numbers; if the second be increased by 2 and the third diminished by 3, the product of the results is the product of the second and third; and, if the first be increased by 3 and the third diminished by 6, the product is that of the first and third. Find the numbers.

13. A person performed a journey of $22\frac{1}{2}$ miles, partly by carriage at 10 miles an hour, partly by train at 36 miles an hour, and the remainder by walking at 4 miles an hour. He did the whole in 1 hour 50 minutes. Had he walked the

first portion, and performed the last by carriage, it would have taken him 2 hours 30½ minutes. Find the respective distances by carriage, train, and walking.

14. A and B start from two places C and D, distant 28 miles, and it is found that A reaches D 3 hours after they meet. Had the distance between C and D been 35 miles, A would have reached a point 28 miles from C 2 hours after he met B. Find the respective rates of A and B.

15. Three trains—a luggage, ordinary, and express—move along three parallel pairs of rails, the listance between the stations being 120 miles. The first two start from the same station, and the express from the opposite. The luggage train, starting 2 hours first, is overtaken by the ordinary in 2 hours; and the express train, starting 1 hour after the ordinary, meets the luggage in 1 hour 7½ minutes. Had all three started from the same station, the ordinary would have been overtaken in 2 hours. Find the respective rates of the trains.

16. If (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) be the respective compositions by weight of three mixtures of three substances, and d_1, d_2, d_3 be the respective prices of the mixtures, find the price per unit of weight of each substance.

17. By alloying two ingots of gold in two given proportions, we form two new ingots of which the fineness of each is known. What is the fineness of each of the given ingots?

18. A group of n persons play as follows:—The first plays with the second and loses $\frac{1}{a}$ of what he has, the second then plays with the third and loses $\frac{1}{a}$ of what he has, the third then with the fourth, losing $\frac{1}{a}$ of what he has, &c., the nth with the first losing $\frac{1}{a}$ of what he has. At the end they each have b. What had each at first?

19. There are three vessels which are respectively $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$ full. If all the liquid is poured into the first vessel it is found to be $\frac{29}{36}$ full; and the capacity of the third is $\frac{8}{10}$ of the sum of the capacities of the first and second; show that their capacities are as the numbers 6, 4, 3.

20. Two waggoners, A and B, start at the same time from two stations on the same road, going respectively 3 and 4 miles an hour. When B has gone $\frac{1}{12}$ of his journey, he finds he must return to A's station; and on meeting A finds him $21\frac{3}{4}$ miles on his way. Having reached A's station, he again turns and reaches his destination along with A; find the distance between the stations, and the length of A's journey.

CHAPTER VII.

QUADRATIC EQUATIONS.

70. Equations of this class, when reduced to a rational integral form, contain the *square* of the unknown quantity, but no higher powers.

When the equation contains the *square only* of the unknown quantity, and *not the first power*, it is called a *pure quadratic*.

Thus, $x^2 - 25 = 0$, $4x^2 + 10 = 19$, $5x^2 = 180$, are *pure quadratics*. When the equation contains the *square* of the unknown quantity, *as well as the first power*, it is called an *adfected quadratic*.

Thus, $x^2 - 5x = 6$, $x^2 - x - 30 = 0$, $2x^2 + x + 3 = 6$, are *adfected quadratics*.

Pure Quadratics.

71. To solve these, we treat them exactly as we do simple equations, until we obtain the value of the *square* of the unknown quantity; then, taking the square root of each side, we obtain the *value* of the unknown quantity. It will be

seen (Art. 35) that the unknown quantity in a pure quadratic has always two values, *equal* in magnitude, but of *opposite* sign.

Ex. 1. Given $3x^2 + 12 = 687$, find x.

We have $3x^2 = 687 - 12 = 675$, or $x^2 = 225$.

$$\therefore x = \pm 15.$$

Ex. 2. Given $\dfrac{2x+7}{2x^2-7x} - \dfrac{2x-7}{2x^2+7x} = \dfrac{56}{6x^2-99}$, find x.

Bringing the fractions on the first side to a common denominator, we have—

$$\frac{(2x+7)^2-(2x-7)^2}{x(4x^2-49)} = \frac{56}{6x^2-99},$$

$$\text{or, } \frac{56x}{x(4x^2-49)} = \frac{56}{6x^2-99},$$

$$\text{or, } \frac{1}{4x^2-49} = \frac{1}{6x^2-99}, \text{ clearing of fractions;}$$

$$\text{then, } 6x^2 - 99 = 4x^2 - 49, \text{ from which}$$

$$x^2 = 25$$

$$x = \pm 5.$$

Adfected Quadratics.

72. *Solution by completing the square.*

Suppose we have given the equation $x^2 + 2ax = 3a^2$, to find x.

It will be remembered that $(x^2 + 2ax + a^2)$ is a *perfect square*, viz., the square of $(x + a)$. It is evident, then, that the first side of our equation will become a perfect square by the addition of a^2 as a third term.

Adding, then, a^2 to each side of the given equation, we have—

$$x^2 + 2ax + a^2 = 4a^2, \text{ or}$$

$$(x+a)^2 = 4a^2.$$

Taking now the square root of each side, we have—

$$x + a = \pm 2a.$$

$$\therefore x = \pm 2a - a.$$

$$= a \text{ or } -3a.$$

We may remark that the quantity a^2, added to the expression $x^2 + 2ax$ in order to make it a *perfect square*, is the square of half the coefficient of x. The operation itself is called *completing the square.*

An adfected quadratic may therefore be solved as follows:—

1. *Reduce and arrange it until all the terms involving* x *are on the first side, and the coefficient of* x^2 *is unity.*

2. COMPLETE THE SQUARE *by adding to each side the* SQUARE OF HALF *the coefficient of* x.

3. *Take the square root of each side, put a double sign to the second side, and transpose the term of the first side not involving* x.

Ex. 1. Solve the equation $3x^2 + 18x + 4 = 52$.

We have $3x^2 + 18x = 52 - 4 = 48$; or, dividing each side by 3, $x^2 + 6x = 16$.

Here, the coefficient of x is 6, the half of which is 3. Adding then the square of 3 to each side, to *complete the square,* we have—

$$x^2 + 6x + 3^2 = 16 + 9 = 25.$$

Taking the square root of each side, we have—

$$x + 3 = \pm 5.$$

$$\therefore x = \pm 5 - 3 = 2 \text{ or } -8.$$

Ex. 2. Given $\dfrac{x+3}{x+4} - \dfrac{x+1}{x+2} = \dfrac{4x+9}{2x+7} - \dfrac{12x+17}{6x+16}$, find x.

We have $\left(1 - \dfrac{1}{x+4}\right) - \left(1 - \dfrac{1}{x+2}\right) = \left(2 - \dfrac{5}{2x+7}\right) - \left(2 - \dfrac{15}{6x+16}\right)$;

or, simplifying, $\dfrac{1}{x+2} - \dfrac{1}{x+4} = \dfrac{15}{6x+16} - \dfrac{5}{2x+7}$;

or, $\dfrac{(x+4)-(x+2)}{(x+2)(x+4)} = \dfrac{15(2x+7) - 5(6x+16)}{(6x+16)(2x+7)}$;

or, simplifying, $\dfrac{2}{x^2+6x+8} = \dfrac{25}{12x^2+74x+112}$;

or, clearing of fractions—

$$24x^2 + 148x + 224 = 25x^2 + 150x + 200;$$

or, transposing and reducing, $-x^2 - 2x = -24$;
or, changing the sign of each side, then—

$$x^2 + 2x = 24;$$

or, completing the square, $x^2 + 2x + 1^2 = 24 + 1 = 25$.

Taking the square root of each side, we have—

$$x + 1 = \pm 5$$

$$\therefore x = \pm 5 - 1 = 4 \text{ or } -6.$$

Ex. 3. Solve the equation $x^2 + 6x + 25 = 0$.

We have $x^2 + 6x = -25$;

or, completing the square—

$$x^2 + 6x + 3^2 = -25 + 9 = -16;$$

or, extracting the square root of each side—

$$x + 3 = \pm \sqrt{-16}$$

$$\therefore x = -3 \pm \sqrt{-16}.$$

As the quantity $\sqrt{-16}$ has no exact or approximate value, the given equation has no real roots. The roots are therefore said to be imaginary.

73. *Solution by breaking into factors.*

We have seen (Art. 30) that it is often easy to find by inspection the factors of quadratic expressions. We may make use of this knowledge to solve quadratic equations.

Ex. 1. Solve the equation $x^2 + 5x = 66$.

Transposing all to the first side, we have—

$$x^2 + 5x - 66 = 0.$$

And, resolving the first side into its elementary factors, we get—

$$(x - 6)(x + 11) = 0.$$

Now, if either of these factors is put equal to 0, the equation is satisfied.

Hence, we have, $x - 6 = 0$, and $x + 11 = 0$;
or, $x = 6$, and $x = -11$.

$\therefore$ 6 and -11 are the roots required.

Ex. 2. Given $x^2 - (a + b)x + ab = 0$, find x.

We have $(x - a)(x - b) = 0$.

Now this equation is satisfied by making either of the factors $= 0$.

$$\left.\begin{aligned}\text{Hence, } x - a = 0, &\text{ or } x = a;\\ \text{and, } x - b = 0, &\text{ or } x = b;\end{aligned}\right\}$$

$\therefore$ a, b are the roots required.

Ex. XX.

1. $4x^2 - 7 = 29.$

2. $5x^2 + 6 = 86.$

3. $\dfrac{3x^2 - 7}{2} = 34.$

4. $\frac{5}{2}x^2 + \frac{3}{8} = 6.$

5. $3x - \dfrac{12}{x} = 0.$

6. $2(a^2 + b^2) - x^2 = (a - b)^2.$

7. *$x\sqrt{24 + x^2} = 4 + x^2.$

8. *$\sqrt{x - a} = \sqrt{x - \sqrt{b^2 + x^2}}.$

9. $\dfrac{ab}{x} + \sqrt{\dfrac{a^2b^2}{x^2} - b^2} = x.$

10. $\dfrac{1}{ax - \sqrt{a^2x^2 - 1}} - \dfrac{1}{ax + \sqrt{a^2x^2 - 1}} = ax.$

11. $\dfrac{\sqrt[3]{a^2 + x^2} + \sqrt[3]{a^2 - x^2}}{\sqrt[3]{a^2 + x^2} - \sqrt[3]{a^2 - x^2}} = b.$

12. $\dfrac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} = a.$

13. $x^2 - 5x + 6 = 0.$

14. $x^2 - x = 72.$

15. $3x^2 - x = 2.$

16. $4x^2 - 9x = 28.$

17. $6x^2 + x - 35 = 0.$

18. $x^2 + 6\frac{3}{5}x + 8 = 0.$

19. $ax^2 + bx + c = 0.$

20. $3x - \dfrac{14}{2x - 3} = 4x - 7.$

21. $x - \dfrac{42}{x} = 1.$

22. $2x + 1 - \dfrac{26}{2x + 1} = 11.$

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* See Remark, page 113.

23. $(a + bx)(cx + d) = (a + b)(c + dx).$

24. $(a - b)(x^2 - b^2) = (a + b)(x - b)^2.$

25. $(a + b)(x - a)(x - b) = abx.$

26. $\frac{a}{x} + \frac{x}{b} = \frac{b}{x} + \frac{x}{a} + \frac{a^2 - b^2}{ab}.$

27. $ax^2 - (c + d)x = bx^2 - \frac{cd}{a - b}.$

28. $a^3x^2 - (a - b)^2 x + a^3 = b^3x^2 + (a^2 + ab + b^2)^2x + b^3.$

29. $\frac{23}{x + 2} - \frac{2}{x} = 3\frac{1}{3}.$

30. $\frac{x + 1}{2x + 3} + \frac{7}{13} = \frac{2x + 5}{4x -}$

31. $\frac{2x + 9}{x} + \frac{x}{2x + 9} = 5\frac{1}{5}.$

32. $\frac{x + 2}{x + 3} + \frac{x - 10}{x + 8} = 0.$

33. $\frac{3x - 7}{4x - 2} + \frac{5x + 3}{7x + 4} = \frac{5}{9}.$

34. $\frac{6x + 5}{2x - 7} + \frac{4x - 1}{x - 2} = \frac{7x + 1}{x - 3}.$

35. $\frac{2x + 1}{x + 2} - \frac{x + 2}{4x + 4} = \frac{7x + 8}{4x + 13}.$

36. $\frac{3x}{x - 2} - \frac{2x + 9}{x + 2} = \frac{2x + 24}{2x - 1}.$

37. $\frac{3x^2 - 2x + 7}{6x^2 - 4x + 11} = \frac{x^2 + 7x - 3}{2x^2 + 14x - 9}$

38. $\frac{3x^2 + 10}{x} - \frac{x^2 + 2x + 3}{x + 2} = \frac{2x^2 + 2x + 10}{x + 1}.$

39. * $\sqrt{5x + 6} + \sqrt{x + 10} = 4.$

40. $\sqrt{3x - 4} - \sqrt{2x - 4} = \sqrt{x}.$

41. * $\sqrt{a^2 + b^2 + x} + \sqrt{a^2 + b^2 - x} = \sqrt{\frac{2ax}{b}}.$

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* See Remark, page 113.

42. * $\sqrt{2a + x} + \sqrt{4a - x} = 2\sqrt{x - a}$.

43. * $\sqrt{ax + b} + \sqrt{bx + a} = \sqrt{(a - b)x + a + b + 2\sqrt{ab}}$.

44. * $\sqrt{x + \frac{b}{a}} + \sqrt{x + \frac{a}{b}} = \left(\sqrt{\frac{x}{a}} + \sqrt{\frac{x}{b}}\right)\sqrt{a + b}$.

45. $\left(\frac{a}{b} - 1\right) . \frac{x - c}{b} = \left(\frac{a^2}{b^2} + 1\right) . \frac{a + b + c}{x} - 2$.

46. $(a - b)(x + b - c)(x + c - a) + a^2(x + b - c) +$
$b^2(x + c - a) + c^2(x + a - b = 0)$.

47. Show that, if x be real, the value of $x + \frac{1}{x}$ cannot lie between 2 and -2.

48. If, in the equation $x - \frac{x^2}{20} = a$, the quantity x be real, show that a cannot be greater than 5.

Equations which may be Solved like Quadratics.

74. Certain equations of a degree higher than the second may be solved like quadratics. It will be seen that, although it is impossible to lay down definite rules for the treatment of every such equation, the object to be attained is either (1.) To throw the equation into the form—

$$X^2 + pX = q,$$

when X is an expression containing the unknown quantity, and solving when possible this equation for X; or (2.), To strike out from each side a factor containing the unknown quantity, thus reducing the dimensions of the equation, and obtaining a value or values of the unknown quantity by equating this factor to zero.

Ex. 1. Solve the equation $x^4 + 144 = 25x^2$.

We have, transposing, $x^4 - 25x^2 = -144$;
or, $(x^2)^2 - 25(x^2) = -144$;

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* See Remark, page 113.

or, completing the square—

$$(x^2)^2 - 25\,(x^2) + \left(\tfrac{25}{2}\right)^2 = \tfrac{625}{4} - 144 = \tfrac{49}{4}.$$

$$\therefore x^2 - \tfrac{25}{2} = \pm\,\tfrac{7}{2}.$$

$$\therefore x^2 = \tfrac{25}{2} \pm \tfrac{7}{2} = 16 \text{ or } 9.$$

$$\text{Hence, } x = \pm\,4 \text{ or } \pm\,3.$$

The given equation has therefore *four* roots, being as many roots as the *degree* of the equation.

Ex. 2. Solve $x^6 + 3\,x^3 = 88$.

We have, $(x^3)^2 + 3\,(x^3) = 88$; or, transposing,

$$(x^3)^2 + 3\,(x^3) - 88 = 0;$$

$$\text{or, } (x^3 - 8)\,(x^3 + 11) = 0.$$

Hence, the given equation is satisfied by—

$$x^3 - 8 = 0, \text{ and also by } x^3 + 11 = 0.$$

We have then $x^3 - 2^3 = 0$, and $x^3 + (\sqrt{11})^3 = 0$; or, breaking into factors, we have—

$$(x - 2)\,(x^2 + x \cdot 2 + 2^2) = 0, \text{ and}$$

$$(x + \sqrt[3]{11})\left\{x^2 - x \cdot \sqrt[3]{11} + (\sqrt[3]{11})^2\right\} = 0.$$

From the *first* of these equations, we get—

$$x - 2 = 0 \text{ or } x = 2,$$

and $x^2 + 2\,x + 4 = 0$, from which two other roots may be found.

And from the *second* equation, we get—

$$x + \sqrt[3]{11} = 0, \text{ or } x = -\sqrt[3]{11},$$

and $x^2 - x\,\sqrt[3]{11} + \sqrt[3]{121} = 0$, which gives two other roots.

We have therefore shown how to obtain the six roots of the given equation.

Ex. 3. Solve—

$$23\,x^2 - 75\,x - 6\,x\sqrt{4\,x^2 - 9\,x + 9} + 40 = 0.$$

Changing the sign of each side, and transposing, we have—

$$6\,x\sqrt{4\,x^2 - 9\,x + 9} = 23\,x^2 - 75\,x + 40;$$

adding to each side $13\,x^2 - 9\,x + 9$, then—

$$13\,x^2 - 9\,x + 9 + 6\,x\sqrt{4\,x^2 - 9\,x + 9} = 36\,x^2 - 84\,x + 49;$$

$$\text{or, } (4\,x^2 - 9\,x + 9) + 2\,(3\,x)\sqrt{4\,x^2 - 9\,x + 9} + (3\,x)^2 = (6\,x)^2 - 2\,(6\,x) \cdot 7 + 7^2;$$

or, $(\sqrt{4x^2 - 9x + 9} + 3x)^2 = (6x - 7)^2$.

$$\therefore \sqrt{4x^2 - 9x + 9} + 3x = \pm (6x - 7), \ldots\ldots\ldots\ldots (1.);$$

or, taking the upper sign—

$$\sqrt{4x^2 - 9x + 9} = (6x - 7) - 3x = 3 - 7.$$

Hence, squaring each side—

$$4x^2 - 9x + 9 = 9x^2 - 42x + 49;$$

$$\text{or, } 5x^2 - 33x + 40 = 0;$$

$$\text{or, } (x - 5)(5x - 8) = 0.$$

$$\therefore x = 5 \text{ or } \tfrac{8}{5}.$$

Again, taking the lower sign of (1.), we have—

$$\sqrt{4x^2 - 9x + 9} = -(6x - 7) - 3x = -9x + 7 \ldots\ldots\ldots (2.);$$

or, squaring—

$$4x^2 - 9x + 9 = 81x^2 - 126x + 49;$$

$$\text{from which } 77x^2 - 117x + 40 = 0;$$

$$\text{or, } (x - 1)(77x - 40) = 0.$$

$$\therefore x = 1 \text{ or } \tfrac{40}{77}.$$

Hence, the roots of the given equation are 5, $\tfrac{8}{5}$, 1, $\tfrac{40}{77}$.

Remark.—If we proceed to verify these values of x, we shall find that the last two values—viz., 1 and $\tfrac{40}{77}$—will not satisfy the given equation unless we obtain the value of $\sqrt{4x^2 - 9x + 9}$ by means of the equation from which these last roots were found.

Thus from (2.) we find, on putting 1 and $\tfrac{40}{77}$ successively for x—

$$\sqrt{4x^2 - 9x + 9} = -9(1) + 7 = -2,$$

$$\text{and } \sqrt{4x^2 - 9x + 9} = -9(\tfrac{40}{77}) + 7 = 2\tfrac{25}{77};$$

and on substituting either of these values of $\sqrt{4x^2 - 9x + 9}$ along with the corresponding value of x, the given equation is satisfied.

Ex. 4. Solve—

$$(x + b + c)(x + b - c)(b + c - x)$$
$$= (a + b + c)(a + b - c)(b + c - a).$$

By inspection we see that a is one of the roots. We shall therefore so arrange the equation as to be able to strike out $x - a$ as a factor of each side (Art. 5).

We have—

$$\frac{(x+b+c)(x+b-c)}{(a+b+c)(a+b-c)} = \frac{b+c-a}{b+c-x};$$

$$\text{or, } \frac{(x+b)^2-c^2}{(a+b)^2-c^2} = \frac{b+c-a}{b+c-x};$$

or, taking the *difference* of numerator and denominator—

$$\frac{(x+b)^2-(a+b)^2}{(a+b)^2-c^2} = \frac{x-a}{b+c-x};$$

$$\text{or, } \frac{(x-a)(x+2b-a)}{(a+b)^2-c^2} = \frac{x-a}{b+c-x}.$$

Dividing each side by $x - a$, we have—

$$\frac{x+2b-a}{(a+b)^2-c^2} = \frac{1}{b+c-x}, \text{ and } x - a = 0, \text{ or } x = a;$$

Hence also $(x + 2b - a)(b + c - x) = (a + b)^2 - c^2$; an ordinary quadratic from which two other roots may be determined.

Ex. 5. * Solve the equation—

$$a^2b^2x^{\frac{1}{q}} - 4a^{\frac{3}{2}}b^{\frac{3}{2}}x^{\frac{p+q}{2pq}} = (a-b)^2x^{\frac{1}{p}}.$$

Dividing each side by $x^{\frac{1}{p}}$, we have—

$$a^2b^2x^{\frac{p-q}{pq}} - 4a^{\frac{3}{2}}b^{\frac{3}{2}}x^{\frac{p-q}{2pq}} = (a-b)^2;$$

$$\text{or, } \left(abx^{\frac{p-q}{2pq}}\right)^2 - 4a^{\frac{1}{2}}b^{\frac{1}{2}}\left(abx^{\frac{p-q}{2pq}}\right) = (a-b)^2;$$

or, completing the square—

$$\left(abx^{\frac{p-q}{2pq}}\right)^2 - 4a^{\frac{1}{2}}b^{\frac{1}{2}}\left(abx^{\frac{p-q}{2pq}}\right) + \left(2a^{\frac{1}{2}}b^{\frac{1}{2}}\right)^2$$
$$= (a-b)^2 + 4ab = (a+b)^2.$$

---

* We shall *assume* in this example that the laws of multiplication and division proved in Arts. 24, 28, hold for fractional indices.

Hence, extracting the square root—

$$abx^{\frac{p-q}{2pq}} - 2a^{\frac{1}{2}}b^{\frac{1}{2}} = \pm(a+b).$$

$$\therefore\ abx^{\frac{p-q}{2pq}} = \pm(a+b) + 2a^{\frac{1}{2}}b^{\frac{1}{2}} = (a^{\frac{1}{2}}+b^{\frac{1}{2}})^2 \text{ or } -(a^{\frac{1}{2}}-b^{\frac{1}{2}})^2;$$

$$\text{or, } x^{\frac{p-q}{2pq}} = \left(\frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}\right)^2 \text{ or } -\left(\frac{a^{\frac{1}{2}}-b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}\right)^2;$$

$$= \left(\frac{1}{b^{\frac{1}{2}}}+\frac{1}{a^{\frac{1}{2}}}\right)^2 \text{ or } -\left(\frac{1}{b^{\frac{1}{2}}}-\frac{1}{a^{\frac{1}{2}}}\right)^2.$$

Then, raising each side to the $(2pq)$th power, we have—

$$x^{p-q} = \left(\frac{1}{b^{\frac{1}{2}}}+\frac{1}{a^{\frac{1}{2}}}\right)^{4pq} \text{ or } \left(\frac{1}{b^{\frac{1}{2}}}-\frac{1}{a^{\frac{1}{2}}}\right)^{4pq}.$$

Hence, taking the $(p-q)$th root, we get—

$$x = \left(\frac{1}{b^{\frac{1}{2}}}+\frac{1}{a^{\frac{1}{2}}}\right)^{\frac{4pq}{p-q}} \text{ or } \left(\frac{1}{b^{\frac{1}{2}}}-\frac{1}{a^{\frac{1}{2}}}\right)^{\frac{4pq}{p-q}}.$$

$$= \left(\frac{1}{b^{\frac{1}{2}}}\pm\frac{1}{a^{\frac{1}{2}}}\right)^{\frac{4pq}{p-q}}.$$

We have also, since the factor $x^{\frac{1}{p}}$ has been struck out—

$$x^{\frac{1}{p}} = 0, \text{ and } \therefore\ x = 0 \text{ as another solution.}$$

Ex. XXI.

1. $4x^4 - 11x^2 = 225.$
2. $5x^6 - 17x^3 = 184.$
3. $x^2 + 5x^{-2} = 25\frac{1}{5}.$
4. $(x^2 + 3x + 3)^2 + 2x^2 = 189 - 6x.$
5. $3x^{-4} - 17x^{-2} = 1450.$
6. $x^n + \dfrac{6}{x^n} = 5.$
7. $\sqrt{x+12} + \sqrt{x+12} = 6.$
8. $\sqrt{x^2+9} + 20 = x^2 + 9.$
9. $\left(\dfrac{12}{x} - x\right)^2 + \dfrac{60}{x} = 6 + 5x.$

10. $(3x + 4) + \sqrt{3x + 4} - 12 = 0.$

11. $x^2 + 2x = \dfrac{2}{x + 4} - 1.$

12. $\dfrac{\sqrt[n]{a - x}}{x} - \dfrac{\sqrt[n]{a - x}}{a} = \dfrac{\sqrt[n]{x}}{a}.$

13. $9x + 24\sqrt{x} - \dfrac{4}{\sqrt{x}}\left(\dfrac{1}{\sqrt{x}} + 9\right) = 65.$

14. $x = \sqrt{a} \cdot \sqrt{b - x} - \sqrt{b} \cdot \sqrt{x + a}.$

15. $\dfrac{1}{(a - \sqrt{a^2 - x^2})} + \dfrac{1}{(a + \sqrt{a^2 - x^2})^2} = \dfrac{a^2 + 2x^2}{x^4}.$

16. $\dfrac{x^2}{b} - 2\sqrt{\dfrac{a^2 - ax + x^2}{b}} = \dfrac{ax}{b} - 1.$

17. $x^{\frac{p+q}{2pq}} - \dfrac{1}{2} \cdot \dfrac{a^2 - b^2}{a^2 + b^2}\left(x^{\frac{1}{p}} + x^{\frac{1}{q}}\right) = 0.$

18. $\sqrt{\dfrac{a^2 + x^2}{x^2}} - \sqrt{\dfrac{x^2}{a^2 + x^2}} = 1.$

19. $(a^{4m} + 1)(x^{\frac{1}{2}} - 1)^2 = 2(x + 1).$

20. $\dfrac{x^2}{x^2 - 9} - \dfrac{975}{256} \cdot \dfrac{x^2 - 9}{x^2} + \dfrac{7}{8} = 0.$

21. $\dfrac{12}{x + 1}\left\{\dfrac{3}{x + 1} + \dfrac{2}{x}\right\} + \dfrac{4}{x^2} = 9.$

22. $6x^2 - 5x - 8\sqrt{3x^2 + 5x - 4} = 12.$

23. $2x^2 + 5x - 2x\sqrt{2x^2 - 7x + 1} = 35.$

24. $20x^2 - 9ax - 8x\sqrt{5x^2 - 3ax + 2a^2} = 7a^2.$

25. $x^2(x - 2)^2 + 6x^2(x - 2) = 24x + 36 - 5x^2.$

26. $2x^2 + 20x - \sqrt{3x^2 - x + 5} = 105.$

27. $4x^2 + 12x - 2x\sqrt{4x^2 - 2x + 19} = 30.$

28. $x^4 + x^2 + 1 = a(x^2 + x + 1).$

29. $x^6 - 1 = 0.$

30. $(x^2 - a)^3 + (x^2 - b)^3 + (x^2 - c)^3$
$= 3(x^2 - a)(x^2 - b)(x^2 - c) + 9ax^4 - a(a + b + c)^2.$

31. $(x - b)(x^2 - c^2) = (a - b)(a^2 - c^2).$

32. $(x - a)(x - b)(x - c) = (m - a)(m - b)(m - c).$

33. $(x - a)(x - b)(x - c) = (a + b)(a + c)(b + c).$

34. $(n - 1)x(x^2 + ax + a^2) = a^3 - x^3.$

35. $(ax - b)^3 + (cx - d)^3 = (a + c)^3x^3 - (b + d)^3.$

36. $x^{2^n} + x^{2^{n-1}} + 1 = a(x^2 + x + 1).$

Simultaneous Quadratic Equations.

75. The following worked examples are given as specimens of the methods to be employed, but it must be understood that practice alone will give the student complete mastery over equations of this class.

$$\text{Ex. 1. Solve } \left.\begin{array}{l} x^2 + y^2 = 20 \dots\dots(1.) \\ x + y = 6 \dots\dots(2.) \end{array}\right\}$$

As we have given the *sum* of the unknown quantities, we shall work for the *difference*.

From (1.), multiplying each side by 2, we have—

$$\begin{array}{lrcl} & 2x^2 \qquad\quad + 2y^2 & = & 40 \\ \text{and from (2.), squaring,} & x^2 + 2xy + y^2 & = & 36 \\ \hline \text{Then, subtracting,} & x^2 - 2xy + y^2 & = & 4; \\ \text{and, taking the square root, we have} & x - y & = & \pm 2 \, (3.) \end{array}$$

$(2.) + (3.)$, then $2x = 8$ or 4, and $\therefore x = 4$ or 2.
$(2.) - (3.)$, then $2y = 4$ or 8, and $\therefore y = 2$ or 4.

NOTE.—Having found that $x = 4$, $y = 2$, we might have told by inspection that the values $x = 2$, $y = 4$, would also satisfy the given equations, for x and y are similarly involved in both equations.

$$\text{Ex. 2. Solve } \left.\begin{array}{l} x^2 - y^2 = 5 \dots\dots(1.) \\ xy = 6 \dots\dots(2.) \end{array}\right\}$$

As we have given here the *difference of the squares* of the unknown quantities, it will be convenient to work for the *sum of the squares.*

From (1.), squaring, $x^4 - 2x^2y^2 + y^4 = 25,$
and from (2.), squaring, &c., $\underline{4x^2y^2 \qquad = 144}$
Then, adding, $x^4 + 2x^2y^2 + y^4 = 169$
and taking the square root, $x^2 + y^2 = \pm 13$ (3.).

(3.) + (1.), then, $2x^2 = 18$ or -8, or $x^2 = 9$ or -4,
and $\therefore x = \pm 3$ or $\pm 2\sqrt{-1}$.

(3.) − (1.), then, $2y^2 = 8$ or -18, or $y^2 = 4$ or -9,
and $\therefore y = \pm 2$ or $\pm 3\sqrt{-1}$.

Note.—The student will see that the pairs of values which satisfy the given equations are, $x = 3, y = 2$; $x = -3, y = -2$; $x = 2\sqrt{-1}$, $y = 3\sqrt{-1}$; $x = -2\sqrt{-1}$, $y = -3\sqrt{-1}$.

Ex. 3. Solve

$$\left.\begin{array}{l} x^2 + y = 11x \ldots\ldots\ldots\ldots\ldots\ldots(1.) \\ y^2 + x = 11y \ldots\ldots\ldots\ldots\ldots\ldots(2.) \end{array}\right\}$$

Subtracting, then, $x^2 - y^2 - x + y = 11x - 11y$;
or, $x^2 - y^2 = 12(x - y)$.

Now $(x - y)$ is a factor of each side, and hence, striking it out, we have—

$x + y = 12 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$(3.),
and also $x - y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$(4.).

Equations (3.) and (4.) may not be used as simultaneous equations, but *each of them* may be used in turn with either of the given equations.

Thus, taking equations (3.) and (1.), we have—

(1.) − (3.), $x^2 - x = 11x - 12$,
from which $x = 6 \pm 2\sqrt{6}$;
and hence from (3.), by substitution, we easily get—

$$y = 6 \mp 2\sqrt{6}.$$

Again, taking equations (4.) and (1.)—

we have, from (4.) $x = y$,
and $\therefore$ from (1.), $x^2 + x = 11x$ or $x^2 = 10x$,
from which $x = 10$ or 0;
and so, from (4.), $y = 10$ or 0.

Hence, the pairs of values satisfying the given equations are—

$$x = 10, y = 10; x = 0, y = 0;$$
$$x = 6 + 2\sqrt{6}, y = 6 - 2\sqrt{6};$$
$$x = 6 - 2\sqrt{6}, y = 6 + 2\sqrt{6}.$$

Note.—It is worth while remarking that when *each of the terms* of the given equations contain at least one of the unknown quantities, the values $x = 0$, $y = 0$ will always satisfy.

Ex. 4. Solve

$$\left.\begin{aligned} 3x^2 - 2xy &= 55 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1.)} \\ x^2 - 5xy + 8y^2 &= 7 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.)} \end{aligned}\right\}$$

Multiplying the equations together *crosswise*, we get—

$$55x^2 - 275xy + 440y^2 = 21x^2 - 14xy;$$

or, transposing, $34x^2 - 261xy + 440y^2 = 0$;

or, $(2x - 5y)(17x - 88y) = 0$,

from which $2x = 5y$, and $17x = 88y$.

Each of these equations taken *in turn* with either (1.) or (2.) will easily give the required values of x and y.

Ex. 5.

$$x^4 + y^4 = 337 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1.)},$$
$$x + y = 7 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.)}$$

From (2.), raising each side to the *fourth power*, we have—

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 2401; \ldots\ldots\ldots \text{(3.)}$$

(3.) − (1.), then $4x^3y + 6x^2y^2 + 4xy^3 = 2064$;

or, $2x^3y + 3x^2y^2 + 2xy^3 = 1032$;

or, arranging, $2xy(x + y)^2 - x^2y^2 = 1032$;

but from (2), $(x + y)^2 = 49$,

and hence, $2xy(49) - x^2y^2 = 1032$;

or, $x^2y^2 - 98xy + 1032 = 0$,

from which $xy = 12 \text{ or } 86, \ldots\ldots \text{(4.)}$

From (2.) and (4.), $x - y$ may now be easily obtained, and hence also the required values of x and y.

Ex. XXII.

1. $x + y = 5$, $xy = 6$.
2. $x - y = 2$, $xy = 15$.
3. $x^2 + y^2 = 25$, $xy = 12$.
4. $x^2 + y^2 = 20$, $x + y = 6$.

5. $x^2 + y^2 = 29,\ x - y = 3.$

6. $x^2 - y^2 = 13,\ (x - y)^2 = 1.$

7. $x^2 - y^2 = 27,\ xy = 18.$

8. $x^2 - y^2 = 12,\ x + y = 6.$

9. $x^2 + y^2 = 53,\ x^2 - y^2 = -45.$

10. $x^2 + xy = 28,\ y^2 + xy = 21.$

11. $x^2 + xy + y^2 = 19,\ xy - x^2 = -3.$

12. $x + y = 13,\ \sqrt{x} + \sqrt{y} = 5.$

13. $x^2 + xy + y^2 = 84,\ x + \sqrt{xy} + y = 14.$

14. $x^3 + y^3 = 35,\ x^2y + xy^2 = 30.$

15. $\frac{1}{x} + \frac{1}{y} = a,\ \frac{1}{x^2} + \frac{1}{y^2} = b.$

16. $x + y = a(x - y),\ x^2 + y^2 = b^2.$

17. $x^4 - y^4 = a,\ x^2 - y^2 = b.$

18. $x + y = 5,\ x^3 + y^3 = 35.$

19. $x + y = 5,\ x^5 + y^5 = 275.$

20. $x^2 + y^2 = \frac{13}{5}(x + y),\ xy = 6.$

21. $x - y = 2,\ x^3 - y^3 = 98.$

22. $xy(x + y) = a,\ x^2y^2(x^2 + y^2) = b.$

23. $xy(x + y) = 30,\ x^2y^2(x^5 + y^5) = 9900.$

24. $4x^2 - 3xy = 18,\ 5y^2 - 2xy = 8.$

25. $x^{\frac{3}{2}} + y^{\frac{3}{2}} = 35,\ x^{\frac{1}{2}} + y^{\frac{1}{2}} = \frac{30}{x^{\frac{1}{2}}y^{\frac{1}{2}}}.$

26. $x^{\frac{3}{2}} + y^{\frac{2}{3}} = 3x,\ x^{\frac{1}{2}} + y^{\frac{1}{3}} = x.$

27. $xy + 6 = \frac{216}{xy},\ x + y + 4 = \frac{77}{x + y}.$

28. $(x + y)^2 + 2(x - y)^2 = 3(x + y)(x - y),\ x^2 + y^2 = 10.$

29. $x^2 + 10xy + y^2 = \frac{73}{5}(x^2 - y^2),\ x^2 + 5y^2 = x + 13y.$

30. $x^4 - 2x^2y^2 + y^4 = 1 + 4xy,\ x^2(x + 1) + y^2(y + 1)$
$= xy.$

31. $3x + y - 9 = \frac{9x^2 - y^2 - 117}{3x - y - 6} = \frac{8x^2 - y^2 + 1}{4x - y + 1}.$

32. $\frac{x+2}{y-2} + \frac{y+2}{x-2} = 5\frac{1}{2}, \frac{x+2}{y+2} + \frac{y-2}{x-2} = 1\frac{5}{6}.$

33. $\frac{\sqrt{x^2-y^2}+(x-y)}{\sqrt{x^2-y^2}-(x-y)} = 3+2\sqrt{2}, \frac{1}{y} - \frac{1}{x} = \frac{2}{xy}.$

34. $y - 2\sqrt{5x^2+y+3} = 32 - 5x^2,$

$\left(\frac{x}{y}+\frac{y}{x}\right)^2 + \left(\frac{x}{y}+\frac{y}{x}\right) = 14\frac{4}{9}.$

35. $\frac{x}{y} - \frac{y}{x} - 3 = -\frac{9}{4}\cdot\frac{xy}{x^2-y^2}, 3x+y=7.$

36. $(x+y)xy = c(bx+ay),$
$xy(bx+ay) = x^2y^2 + abc(x+y-c).$

37. $y^4 = x^2(ay-bx), x^2 = ax - by.$

38. $\frac{\sqrt{y^2+1}+1}{y} = \frac{\sqrt{x+9}+3}{\sqrt{x}}, x(y+1)^2 = 36\left(y^3 + \frac{1}{y^6}\right).$

39. $x = \frac{a}{y+z}, y = \frac{b}{x+z}, z = \frac{c}{x+y}.$

40. $x+y+z=6, xy+xz+yz=11, xyz=6.$

41. $x^2 - yz = 1, y^2 - xz = 0, z^2 - xy = 0.$

42. $xyz = a^2(x+y) = b^2(y+z) = c^2(x+z).$

43. $x^2+y^2+z^2 = \frac{a^2}{x^2} = \frac{b^2}{y^2} = \frac{c^2}{z^2}.$

44. $x+y+z+u = 4a+4b,$
$xy+xz+xu+yz+yu+zu = 6a^2+12ab+6b^2,$
$xyz+xyu+xzu+yzu = 4a^3+12a^2b+12ab^2+4b^3,$
$xyzu = a^4+4a^3b+6a^2b^2+4ab^3+b^4.$

45. $x^2y^2 + xy^2z + x^2yz = a,$
$y^2z^2 + xy^2z + xyz^2 = b,$
$x^2z^2 + x^2yz + xyz^2 = c.$

46. $(x+y)^3 + z^3 = 1125,$
$x+y+z = 15,$
$xy = 24.$

47. If $ax^3 = by^3 = cz^3$, and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = a$, show that $ax^2+by^2+cz^2 = (a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}})^3a^2.$

48. Given $R = 1 + r$, $P = \frac{A}{r}(1 - R^{-n})$, $M = PR^n$, show that $R = A\left(\frac{1}{P} - \frac{1}{M}\right) + 1$.

CHAPTER VIII.

Problems Producing Quadratic Equations.

76. We shall now discuss one or two problems whose solutions depend upon quadratic equations.

Ex. 1. A person raised his goods a certain rate per cent., and found that to bring them back to the original price he must lower them $3\frac{1}{3}$ less per cent. than he had raised them. Find the original rise per cent.

Let x = the original rise per cent.,

then $\frac{x}{100 + x} \cdot 100$ = the *fall per cent.* to bring them to the original price.

Hence, by problem—

$$x - \frac{100\,x}{100 + x} = 3\tfrac{1}{3}, \text{ which solved, gives}$$

$$x = 20 \text{ or } - 16\tfrac{2}{3}.$$

The value $x = 20$ is alone applicable to the problem. Remembering, however, the algebraical meaning of the *negative* sign, it is easy to see that the second value, $x = - 16\frac{2}{3}$, gives us the solution of the following problem:—

A person *lowered* his goods a certain rate per cent., and found that to bring them back to the original price he must *raise* them $3\frac{1}{3}$ more per cent. than he had *lowered* them. Find the original fall per cent.

The above solution tells us that the fall required is $16\frac{2}{3}$ per cent.

Had we worked the latter problem first, we should have obtained $x = 16\frac{2}{3}$ or $- 20$, the value $x = - 20$ indicating the solution of the former problem.

Ex. 2. Find a number such that when multiplied by its deficiency from 100 the product is 196.

Let x = the number,
then $100 - x$ = its deficiency from 100.
Hence, by problem—

$$x(100 - x) = 196, \text{ or } x^2 - 100x + 196 = 0;$$

from which, $x = 2$ or 98.

Both these values will be found to be consistent with the conditions of the problem.

Ex. 3. The number of men required to build a house is such that, when four times the number is subtracted from three times the square of the number, the result is 160. Find the number of men.

Let x = the number of men,
then, by problem—

$$3x^2 - 4x = 160, \text{ from which}$$
$$x = 8 \text{ or } - 6\tfrac{2}{3}.$$

The value $x = 8$ is alone applicable to the problem as it stands. If, however, we may conceive of a *fractional* number of men—and this we may easily do here by supposing a *boy's* work to be equal to $\frac{2}{3}$ of a man's—we find that the second result gives us the solution of the following problem:—

The number of men required to build a house is such that when four times the number is *added to* three times the square of the number, the result is 160. Find the number.

The answer, as above indicated, is 6 men and 1 boy, where a boy is worth $\frac{2}{3}$ of a man.

The student will find, however, that in some cases there is no *obvious* interpretation to the second result, owing occasionally to the fact that certain *terms* are used in the problem to which the results will not apply, and indeed that the algebraical expression of the conditions of the problem is more general than the language of the problem itself.

Ex. XXIII.

1. Find a number whose square is equal to the product of two other numbers, one of which is less by 6 than the required number, and the other greater by 9 than twice that number.

2. When the numerator and denominator of a certain fraction are each increased by unity the fraction is increased by $\frac{1}{190}$, and when they are each diminished by unity the fraction is diminished by $\frac{1}{95}$. Find the fraction.

3. The mean proportional between the excess of a certain number above 21, and its defect from 37, is 28. Find the number.

4. A number of articles, which were bought for £4, cost each 3 shillings more than half as many shillings as there were articles. Find the number of articles.

5. There is a square court-yard, such that if its length be increased by 10 feet, and its breadth diminished by 20 feet, its area would be 5,104 feet. Required the side of the square.

6. If the number of shillings given for an article be added to the number of articles which can be bought at the same price for 18 shillings, the result is 11. Find the price.

7. Two travellers set out to meet each other from two places 180 miles distant; the first goes 3 miles an hour, and the second goes 1 mile more per hour than one-fourth of the number of hours before they meet. When will they meet?

8. A farmer bought a number of calves, sheep, and pigs, the number of calves being equal to that of the sheep and pigs together. For the calves he gave 64s. a head, and for the sheep twice as many shillings as there were sheep. He paid £153. 12s. for the calves and sheep together, and £36. 12s. for the pigs—a pig costing as much as a sheep and calf together. Find the cost of the sheep per head.

9. There are two squares, and an oblong whose sides are equal to those of the squares, and it is noted that three times the area of the first square exceeds four times the area of the oblong by 3 square feet, while twice the area of the square, together with three times the area of the rectangle, is 36 square feet. Required the sides of the squares.

10. The sum of two quantities is equal to 6 times the square of their product, and the sum of their cubes is equal to 36 times the product of their fifth powers. Find the quantities.

11. The solid content of a rectangular parallelopiped is 60 cubic feet, and the total area of the sides is 94 square feet, while the sum of the edges is 48 feet. Required the dimensions.

12. The products of the number of units of length in the sides of a polygon of n sides, when taken $n - 1$ together are respectively a_1, a_2, a_3, &c., a_n. Required the lengths of the sides.

13. A, B, and C can together do a piece of work in a day, and C's rate of work is the product of the rates of A and B. Moreover, C is one-fifth as good a workman as A and B together. Find the respective times required for A, B, C to do a piece of work.

14. The compound interest of a certain sum of money for 3 years is a, and the third year's interest is b. Find the principal and the rate per cent.

15. A owes B £a due m months hence, and also £b due n months hence. Find the equated time, reckoning interest at 5 per cent. per annum.

16. Find three quantities such that the sum of any two is equal to the reciprocal of the third.

17. Find three magnitudes, when the quotients arising from dividing the products of every two by the other are respectively a, b, c.

CHAPTER IX.

THE THEORY OF QUADRATIC EQUATIONS AND EXPRESSIONS.

77. *A quadratic equation cannot have more than two roots.*

It has been shown (Art. 30) that $f(x)$ is divisible by $x - a$, when it vanishes on putting a for x; that is, when a is a root of the equation $f(x) = 0$.

Hence, if α, β are roots of the quadratic equation

$$x^2 + px + q = 0,$$

$x - \alpha$ and $x - \beta$ are each of them factors of the expression $x^2 + px + q$, whatever be the value of x.

Moreover, $x - \alpha$ and $x - \beta$ are the only linear factors of $x^2 + px + q$, since their product gives an expression of two dimensions whose first term is x^2, the same as that of the expression $x^2 + px + q$. We therefore have identically

$$x^2 + px + q = (x - \alpha)(x - \beta),$$

Hence no value of x can make $x^2 + px + q$ vanish unless it make one of the factors $x - \alpha$, $x - \beta$ vanish; that is, $x^2 + px + q$ cannot vanish for any other values of x than α and β.

It follows, therefore, that α, β are the only roots of the equation

$$x^2 + px + q = 0.$$

78. *If* α, β *are the roots of the equation* $x^2 + px + q = 0$, *then* $\alpha + \beta = -p$, *and* $\alpha\beta = q$.

For (Art. 77) we have *identically*,

$$\begin{aligned} x^2 + px + q &= (x - \alpha)(x - \beta), \text{ or,} \\ &= x^2 - (\alpha + \beta)x + \alpha\beta. \end{aligned}$$

Here we have the coefficient p on the first side represented by $-(\alpha + \beta)$ on the second side, and the constant term q represented by $\alpha\beta$.

We then have

$$\begin{aligned} -(\alpha + \beta) = p, \text{ or } \alpha + \beta &= -p, \\ \text{and } \alpha\beta &= q. \quad Q.E.D. \end{aligned}$$

COR. 1. It easily follows that

(1.) $\alpha^2 + \beta^2 = p^2 - 2q$.

(2.) $\alpha - \beta = \sqrt{p^2 - 4q}$.

(3.) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha + \beta}{\alpha\beta} = -\dfrac{p}{q}$.

COR. 2. If the given equation be of the form $ax^2 + bx + c = 0$, we may write it thus:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

and we have

(1.) $\alpha + \beta = -\dfrac{b}{a}$.

(2.) $\alpha\beta = \dfrac{c}{a}$.

(3.) $\alpha^2 + \beta^2 = \dfrac{b^2 - 2ac}{a^2}$

(4.) $$\alpha - \beta = \frac{1}{a}\sqrt{b^2 - 4ac}.$$

(5.) $$\frac{1}{\alpha} + \frac{1}{\beta} = -\frac{b}{c}.$$

79. *The quadratic expression* $ax^2 + bx + c$ *is a perfect square when* $b^2 = 4ac$.

When $ax^2 + bx + c$ is a perfect square, its *linear* factors must be identical; and hence the condition is that the equation

$$ax^2 + bx + c = 0$$

must have *equal roots.*

Solving this equation we get

$$x = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac}).$$

Now these two values of x can be equal only when

$$b^2 - 4ac = 0, \text{ or when } b^2 = 4ac.$$

Hence the condition that $ax^2 + bx + c$ shall be a perfect square is that $b^2 = 4ac$.

COR. The roots of the equation $ax^2 + bx + c = 0$ are *rational* when $b^2 - 4ac$ *is a perfect square,* and are *impossible* when $b^2 < 4ac$.

80. *To investigate the relations between the coefficients* A, B, C, a, b, c, *in order that the expression*

$$Ax^2 + By^2 + Cz^2 + ayz + bxz + cxy$$

shall be a perfect square.

We must have *identically*

$$\begin{aligned} &Ax^2 + By^2 + Cz^2 + ayz + bxz + cxy \\ &= (\sqrt{A}\,.\,x + \sqrt{B}\,.\,y + \sqrt{C}\,.\,z)^2 \\ &= Ax^2 + By^2 + Cz^2 + 2\sqrt{BC}\,.\,yz + 2\sqrt{AC}\,.\,xz \\ &\quad + 2\sqrt{AB}\,.\,xy. \end{aligned}$$

Equating the coefficients of identical terms, we have

$$a = 2\sqrt{BC}, \quad b = 2\sqrt{AC}, \quad c = 2\sqrt{AB},$$

the relations required.

COR. Solving for A, B, C, we may put these relations in the following form:

$$A = \frac{bc}{2a},\ B = \frac{ac}{2b},\ C = \frac{ab}{2c}.$$

81. *The values of* x *and* y *derived from the equations*

$$ax + by = \frac{a^3}{x} + \frac{b^3}{y} = c^2$$

are rational, when $a^2 + b^2 = c^2$.

For, solving for x, we get—

$$ac^2x^2 + (b^4 - c^4 - a^4)x + a^3c^2 = 0.$$

Hence (Art. 79, Cor.) the values of x are rational when $(b^4 - c^4 - a^4)^2 - 4\,ac^2 \,.\, a^3c^2$ is a perfect square.

The values of x are therefore rational when

$$(b^4 - c^4 - a^4)^2 = 4\,a^4c^4, \text{ or}$$
$$b^4 - c^4 - a^4 = \pm\, 2\,a^2c^2, \text{ or}$$
$$(a^2 \pm c^2)^2 = b^4, \text{ or when } a^2 \pm c^2 = \pm\, b^2.$$

And we may similarly show that the values of y are rational when

$$b^2 \pm c^2 = \pm\, a^2.$$

Now, on examination it will be found that the only condition common to these two sets is that

$$a^2 + b^2 = c^2.$$

Hence the values of x and y are both rational when

$$a^2 + b^2 = c^2. \quad Q.E.D.$$

82. *To investigate the condition that the expression*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

is resolvable into elementary factors.

If the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ be solved either for x or y, the condition that either can be expressed in terms of the other in a rational form will (Art. 1) be the condition that the given expression can be broken up into elementary rational factors.

Solving for x, we find x equal to

$$= \frac{By + D \pm \sqrt{(B^2 - 4AC)y^2 + 2(BD - 2AE)y + (D^2 - 4AF)}}{2A}.$$

Now, in order that this may be rational, we must have

$$(B^2 - 4AC)y^2 + 2(BD - 2AE)y + (D^2 - 4AF)$$

a perfect square.

Hence (Art. 79) we must have

$$\{2(BD - 2AE)\}^2 = 4(B^2 - 4AC)(D^2 - 4AF)$$

or $$(BD - 2AE)^2 = (B^2 - 4AC)(D^2 - 4AF) \quad \dots (1).$$

This may be transformed into either of the following forms:

$$AE^2 + CD^2 + FB^2 - BDE - 4ACF = 0 \quad \dots\dots\dots (2).$$
$$(BE - 2CD)^2 = (B^2 - 4AC)(E^2 - 4CF) \quad \dots\dots\dots (3).$$
$$(DE - 2BF)^2 = (D^2 - 4AF)(E^2 - 4CF) \quad \dots\dots\dots (4).$$

Ex. XXIV.

1. Find the difference of the squares of the roots of the equation $3x^2 - 12x - 43 = 0$.

2. Construct an equation whose roots are the reciprocals of the roots of $ax^2 + bx + c = 0$.

3. If α, β are the roots of the equation $x^2 - px + q = 0$, construct the equation whose roots are $\alpha - \gamma$, $\beta - \gamma$.

4. Find the value of a when the equation

$$ax^2 - 36x + 81 = 0$$

has equal roots.

5. Find the relation between the quantities a, b, c, a', b', c', so that the equations

$$ax^2 + bx + c = 0, \text{ and } a'x^2 + b'x + c' = 0$$

shall have a common root.

6. The equations $ax^2 + bx + c = 0$, and $2ax + b = 0$, have common roots when the former has two equal roots.

7. When is the equation

$$\sqrt{m^2x + a^2} + \sqrt{n^2x + b^2} + \sqrt{p^2x + c^2} = 0$$

reducible to a simple equation?

8. If α, β are the roots of the equation $x^2 - px + q = 0$, form the equation whose roots are $\alpha + \beta$, and $\alpha - \beta$.

9. If $x^2 + px + q$ and $x^2 + mx + n$ have a common factor, show that this common factor is $x + \dfrac{n - q}{m - p}$.

10. Hence show that the following relation holds, viz.,

$$(n - q)^2 + n(m - p)^2 = m(m - p)(n - q).$$

11. Show that the values of x and y derived from the equations $ax^2 + by^2 = a^2x + b^2y = c^3$ are rational when $a^3 + b^3 = c^3$.

12. Show that the expression $6x^2 + 17xy + 12y^2 + 4x + 7y - 10$ can be broken into elementary factors, and determine the factors.

ANSWERS.

I.—Page 13.

1. $8, 27, 6, -16$. **2.** $6, 2, 4, 0$. **3.** $5, 11, -7$.
4. $11, 16$. **5.** $-2, -15$. **6.** 8. **7.** $-5, -7$.
8. $-11, -3$. **9.** $7, -69$. **10.** $72, -68$.
11. $-2, -8$. **12.** $-10, 32$.

II.—Page 18.

1. $51, 6, 3$. **2.** $125, 35$. **3.** $-513, -65$.
4. $-1224, 30$. **5.** $0, 2$. **6.** $1, -27$. **7.** $2, -2$.
8. $1591, 0$. **9.** 144. **10.** 1521.
11. 145. **12.** $-\dfrac{37}{\sqrt[3]{1029}}$.

III.—Page 21.

1. $10a + 3b$. **2.** $11a^2$. **3.** $12b + 8c$. **4.** 0.
5. $8a^2 + ab$. **6.** $3x^4 - x^2 - 15x - 2$. **7.** $4ab - 4$.
8. $x^3 + y^3 + z^3 - 3xyz$. **9.** $x^4 + x^2y^2 + y^4$.
10. $a^3 + b^3 + c^3 + 3a^2b + 3ab^2 - a^2c - ac^2 - b^2c - bc^2 - 2abc$.
11. $x^4 + y^4 + z^4 - 4x^3y + 4x^3z - 4xy^3 - 4y^3z + 4xz^3 - 4yz^3 + 6x^2y^2 + 6x^2z^2 + 6y^2z^2 - 12x^2yz + 12xy^2z - 12xyz^2$. **12.** 0.

IV.—Page 22.

1. $4a + 2b + 5c$. **2.** $-4x + 2y - 6z$.
3. $a^2 - 3ab - b^2 - 5a - 7b - 8$.

4. $2a^4 - 2a^2x^2 + x^4$. 5. $4a^4 + 8a^2b^2 + 4b^4$.
6. $-3x^3 - 3z^3 - 3x^2z - 3xz^2$.
7. $x^4 - ax^3 - 9a^2x^2 - 3a^3x - 2a^4$. 8. 0.
9. $-a - b - c - d + e + f + g + h$.
10. $2x^4 - 8x^3y + 12x^2y^2 - 8xy^3 + 2y^4$.
11. $x^4 - 2x^2y^2 + y^4$.
12. $a^2 + b^2 - 2c^2 + 2ab - 2ac - 2bc$.

V.—Page 24.

1. $-x + 8y + z$. 2. $a^3 - b^3$. 3. $13 - 6x$.
4. $5x^2 - 3x - 7$.
5. $(a + b) + (c - d)$, $a - (b - c + d)$,
$\left\{a - (b - c)\right\} - d$.
6. $-(6a - 7b) - (3c - 5d)$, $-6a + (7b - 3c + 5d)$,
$-\left\{6a - (7b - 3c)\right\} + 5d$.
7. $-(4x^3 - 12x^2y) - (12xy^2 - 4y^3)$,
$-4x^3 + (12x^2y - 12xy^2 + 4y^3)$,
$-\left\{4x^3 - (12x^2y - 12xy^2)\right\} + 4y^3$.
8. $(a^3 - b^3) - (c^3 - 3abc)$, $a^3 - (b^3 + c^3 - 3abc)$,
$\left\{a^3 - (b^3 + c^3)\right\} + 3abc$.
9. $-(a - b + d)x^2 - (a - b - 3c)xy$
$-(2b - c - e - f)y^2$.
10. $(a - b + c - d)x - (c - d + e - f)y$
$-(e - f + g - h)z$.
11. $(a - b - d)x - (a - 7b - c)y - (b - c + d)z$.
12. $(z - x)a^2 + (x - y)ab + (y - z)b^2$.

VI.—Page 27.

1. $12a^2 - ab - 6b^2$, $18x^2 - 9xy - 35y^2$.
2. $x^3 - 6xy^2 + 4y^3$, $30x^3 + 49x^2y + 9xy^2 - 4y^3$.
3. $a^4 - 2a^2b^2 + b^4$, $a^4 - b^4$. 4. $x^4 - y^4$, $x^3 - y^3$.
5. $a^3 + b^3 + c^3 - 3abc$. 6. $x^8 - y^8$.
7. $a^3 + 3a^2b + 3ab^2 + b^3$.
8. $5a^5 + 5a^4 - 405a - 405$.

9. $-7x^3 + 17x^2 - 5x - 2$, $a^3 - x^3$.

10. $a^8 + 2a^6b^2 + 3a^4b^4 + 2a^2b^6 + b^8$.

11. $af + (ac + bf)x + (b + f)cx^2 + (c^2 + df)x^3 + cdx^4$.

12. $x^3 + (p - a)x^2 + (q - ap)x - aq$.

13. $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$.

VII.—Page 32.

1. $x^2 - 2xy + y^2$, $9a^2 - 30ab + 25b^2$, $16c^4 + 8c^2d^2 + d^4$, $9x^4 - 12x^2y^2 + 4y^4$.

2. $a^8 - b^8$.   3. $m^2x^2 - n^2y^2$, $50a^4 - 18b^4$.

4. $(a + c)^2 - (b + d)^2$, $(a + b)^2 - (c + d)^2$.

6. $x^2 + 4x - 5$, $x^2 + 7x + 10$, $x^2 + 2x - 15$, $x^2 - 25$.

7. $x^4 - 37x^2 - 24x + 180$.

8. $\frac{1}{16}\left\{4a^2b^2 - (a^2 - b^2 - c^2)^2\right\}$.

9. $4(a^2 + b^2 + c^2 + d^2)$.

VIII.—Page 38.

1. $4a^2 - ab + 2b^2$, $xy - 4y^2$.

2. $2a^4 - 5a^3b + \frac{20}{3}a^2b^2$, $2x^2 + 3xy + 2y^2$.

3. $ax^{m-n} + bx^{-n}y^n + cx^{-(m+n)}y^{2n}$,
$ax^{m-n} + by^n + cx^{n-m}y^{2n}$.

4. $6x + 4y$, $5x - 3y$.   5. $1 + x + x^2$.

6. $-3x - 4$, $-7x + 3$.

7. $x^3 - 3x^2y + 5xy^2 + 27y^3 + \dfrac{72y^4}{x - 3y}$,
$x^3 - x^2y - 3xy^2 + 15y^3 - \dfrac{24y^4}{x + y}$.

8. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$, $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

9. $ax^m + by^n$.   10. $a^3 + a^2b + ab^2 + b^3$.

11. $b + c$, $a + b$.   12. $a + bx + cx^2$.

13. $a^4 - (p - 1)a^3 - (p - q - 1)a^2 - (p - 1)a + 1$.

14. The given expression is—
$(a + b + c - d)(a + b - c + d)(a - b + c + d)(a - b - c - d)$.

16. $-x^4y^2 - x^2y^4,\ x + x^{-1}$.
17. $(x+y)^3 + (x+y)^2z - (x+y)z^2 - z^3$.
20. $(a-c)^2 - 2(a-c)(b-d) + (b-d)^2$.

IX.—PAGE 45.

1. $(x+3a)(x-3a),\ (4y^2+5z^2)(4y^2-5z^2)$,
$6(2a+3b)(2a-3b),\ (2x-3y)(4x^2+6xy+9y^2)$.
2. $x(x-y)(x^2+xy+y^2)$,
$(a-b)(a+b)(a^2+b^2)(a^4+b^4)$,
$xy(x+y)(x^2-xy+y^2)$,
$2xy^2z(x+2z)(x-2z)$.
3. $(a^2-2b^2)(a^2+2b^2),\ (x^2+xy+y^2)(x^2-xy+y^2)$,
$(a+b)^2(a-b)^2(a+b+c)(a+b-c)$.
4. $(a+b+c+d)(a+b-c-d)$,
$(a+b-c+d)(a-b+c+d)$,
$(a+b-c)(a-b+c)$.
5. $5(2x+9),\ 3(2x+7),\ (3a+b-c)(a+b-c)$.
6. $(x^2+xy+y^2)\left\{(x-y)(x^3-y^3)+4x^2y^2(x^2-xy-y^2)\right\}$,
$(x^2+xy+y^2)(x^2-xy+y^2)(x+y)^2(x-y)^2$.
7. $(x-10)(x+7),\ (x+1)(x+10)$,
$(a-7b)(a-8b),\ (x-16)(x+12)$.
8. $(ax+7by)(ax-6by),\ 3a(x-10)(x+2)$,
$ac(c-8)(c+3)$.
9. $(3x+5)(2x-7),\ (2x+9)(4x-15)$,
$3(2x+3)(3x-8),\ (4x-7)(5x+6)$.
10. $xy(3x+y)(x+3y),\ x(5x+3a)(4x+5b)$,
$(mx+p)(nx+q)$.
11. $x^3+2x^2+4x+8,\ 3x^4-6x^3+12x^2-24x+48$,
x^4+3x^2+9.
12. $(a+b)^2-(a+b)(c+d)+(c+d)^2,\ a-b+c+d$.
13. $a^4+pa^3+qa^2+ra+s,\ 2a^4$. 14. $-102,\ 17$.

X.—PAGE 48.

1. $a^4b^6,\ -27a^3b^9,\ 16a^8b^2c^2,\ -x^9y^2z^6$.

2. $a^4 + 12\,a^3b + 54\,a^2b^2 + 108\,ab^3 + 81\,b^4$,
$16\,a^4 + 32\,a^3b + 24\,a^2b^2 + 8\,ab^3 + b^4$,
$a^5 - 5\,a^4b + 10\,a^3b^2 - 10\,a^2b^3 + 5\,ab^4 - b^5$,
$27\,a^3 - 108\,a^2b + 144\,ab^2 - 64\,b^3$.

3. $16\,m^4 + 32\,m^3 + 24\,m^2 + 8\,m + 1$,
$125\,x^3 + 150\,x^2 + 60\,x + 8$,
$81\,a^4 - 432\,a^3c + 864\,a^2c^2 - 768\,ac^3 + 256\,c^4$,
$-\,a^3 - 3\,a^2b - 3\,ab^2 - b^3$.

4. $x^4 + 2\,x^3 + 3\,x^2 + 2\,x + 1$,
$9\,a^2 + b^2 + 16\,c^2 + d^2 - 6\,ab + 24\,ac - 6\,ad - 8\,bc$
$+ 2\,bd - 8\,cd$,
$a^2 + 4\,b^2 + c^2 + 4\,ab - 2\,ac - 4\,bc$,
$9\,(a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc)$.

5. $1 + 3\,x - 5\,x^3 + 3\,x^5 - x^6$, $(ax + by)^3 + 3\,(ax + by)^2\,cz$ + &c.

6. $1 + 7\,x + 21\,x^2 + 35\,x^3 + 35\,x^4 + 21\,x^5 + 7\,x^6 + x^7$,
$(1 + x)^4 + 4\,(1 + x)^3x^2 + 6\,(1 + x)^2x^4 + 4\,(1 + x)x^6 + x^8$,
$(a + bx)^4 + 4\,(a + bx)^3\,cx^2$ + &c.

7. $a^8x^8 - 8\,a^7bx^7y + 28\,a^6b^2x^6y^2 - 56\,a^5b^3x^5y^3 + 70\,a^4b^4x^4y^4$
$- 56\,a^3b^5x^3y^5 + 28\,a^2b^2x^2y^6 - 8\,ab^2xy^2 + b^8y^8$,
$729\,x^6 - 243\,x^4y^2 + 27\,x^2y^4 - y^6$,
$x^9 - 3\,x^6y^3 + 3\,x^3y^6 - y^9$.

8. $x^{12} - 2\,x^6y^6 + y^{12}$,
$a^{12} - 6\,a^{10}b^2 + 15\,a^8b^4 - 20\,a^6b^6 + 15\,a^4b^8 - 6\,a^2b^{10} + b^{12}$.

9. $a^3 + 3\,a^2b + 3\,ab^2 + b^3$. 10. $(a - c)^3$

11 $2\,(1 + 3\,x^2)^4$. 12. $81\,x^4y^4\,(x + y)$.

XI.—Page 62.

1. $2\,xy^2z^3$, $4\,a^2y^2$, $x^2 + a^2$. 2. $2\,x^3 - 3\,x^2y + 5\,xy^2$.
3. $5\,a^2 - 3\,ab + b^2$. 4. $1 - 2\,x + 3\,x^2 - 4\,x^3$.
5. $a + bx + cx^2 + dx^3$. 6. $a^2x^n - 3\,ax^{n-1} + 4\,x^{n-2}$.
7. $x + x^{-1}$, $ax^{-1} - a^{-1}x$ 8. $3\,x^m \dot{-} \frac{a^2}{2}\,5\,a$.
9. 36, 79, 207, 289. 10. $\dot{1}03{\cdot}2$, $\cdot 024$, $\cdot\dot{3}$, $\frac{4}{9}$.
11. 4·123, 1·224, ·618, 3·732.
12. ·0203, $4{\cdot}\dot{6}$, 3·5036. 13. $2\,ab^2y^4$, $5\,x^4y$, $a + 2\,b$.

14. $x^4 + 3x^2 - 7$. 15. $x + y - c$.
16. $x + x^{-1}$, $xy^{-1} + 1$. 17. 18018, 11·11.
18. 2·73, $\cdot\dot{3}$, $\frac{3}{7}$. 19. ·5, ·2599.
20. $\sqrt[3]{5} = 1{\cdot}709$. 21. 7·6457, 50·2487.
22. 4·4142, 3·7320. 23. ·25. 24. 0.

XII.—Page 69.

1. $x - 3$. 2. $x^2 + 4x + 3$. 3. $x + 3$.
4. $x + a$. 5. $a^2 + ab + b^2$. 6. $x - 1$.
7. $x - 3$. 8. $3x - 2$. 9. $12x^2 + 5x$.
10. $3a + 2b$. 11. $x - 1$. 12. $2x^3 - 4x^2 + x - 1$.
13. $2a + 3b + c$. 14. $x^2 + 1$. 15. $x - 1$.
16. $2x^2 - 3x + 9$. 17. 1. 18. $a + b + c$.

XIII.—Page 72.

1. $12a^3x^3y^3$. 2. $60a^2b^2c^2$.
3. $(a - b)(b - c)(c - a)$. 4. $a^2x(x^2 - a^2)$.
5. $(x + 1)(x + 2)(x + 3)$. 6. $(x - 6)(x + 5)(x - 5)$.
7. $(2x + 7)(3x + 8)(4x + 5)$.
8. $210(x^2 + 1)(x^3 + 1)$. 9. $a^2(x^4 + a^2x^2 + a^4)$.
10. $(x + a)(x + b)(x + c)$. 11. $1 - x^{16}$.
12. $(x + 1)(x + 2)(x + 3)(x - 5)(x - 6)(x + 5)$.
13. $a(a - b)^3(a^2 - ab + b^2)$.
14. $(x - 1)(x + 1)(x^2 + 1)(6x^3 + 5x^2 + 2x - 1)$.
15. $(a^2 - b^2)^4$.
16. $2x(x^2 - 1)(3x^2 + 3x - 1)(2x^2 + 2x - 5)(2x^2 - 2x + 5)$.
17. $3(x - 2)^2(x + 4)(5x^3 + 10x^2 + 20x + 18)$.
18. $a^2b^2(a + b)(a - b)$. 19. $15(3x - 10)(x^4 - 16)$.
20. $(x^2 - y^2)^2(x^2 + y^2)(x^4 + y^4)$. 21. $x^6 - a^6$.
22. $(a + b + c + d)(a + b - c - d)(a - b - c + d)$.
23. $(a + b + c)(a^3 + b^3 + c^3 - 3abc)$.
24. $(a + b + c + d)(b + c + d - a)(a + c + d - b)$
$(a + b + d - c)(a + b + c - d)(a + b - c - d)$.

XIV.—Page 75.

1. $\dfrac{x - 1}{x + 6}$, $\dfrac{x^2 + x - 2}{x^2 + 5x + 5}$.

2. $\dfrac{3x + 7}{7x + 2}$, $\dfrac{2x^2 + 2x + 9}{5x^2 + 2x - 2}$.

3. $\dfrac{(a - b)(a + b)^2}{a^2 - 3ab + b^2}$, $\dfrac{4a^2 + b^2}{a + 3b}$.

4. $\dfrac{x(y - z) - y^2}{x(y + z) - y^2}$, $\dfrac{1}{x^2 - y^2}$. 5. $\dfrac{2a}{a^2 - b^2}$, $\dfrac{2b}{a^2 - b^2}$.

6. 0. 7. $\dfrac{x^3 - x^2 + 3x - 1}{2(x - 1)(x^2 + 1)^2}$.

8. $\dfrac{10}{(x + 1)(x + 2)^2(x^2 + 1)}$.

9. $\dfrac{8x - 20}{(x + 1)^2(x + 3)}$. 10. $\dfrac{8x^2 + 8}{(x - 1)^4(x^3 + 1)}$.

11. 0. 12. 0. 13. 1. 14. $a + b + c$. 15. 1. 16. 0.

17. $-\dfrac{16a^5x}{(a^4 - x^4)^2}$. 18. $\dfrac{a^2 - ab + b^2}{a^2 + ab - b^2}$. 19. 1. 20. 1.

21. $\dfrac{8}{(x - y)(y - z)(x - z)}$. 22. $\dfrac{1}{x^3}$.

23. $\dfrac{1}{(x - a)(x + b)(x + c)}$. 24. $\dfrac{5a^2}{x} - 4a + 3x$.

25. 1. 26. $\dfrac{4a^2b^2}{(a^2 - b^2)^2}$. 27. $\left(\dfrac{a}{a + b}\right)^2 + \left(\dfrac{b}{a - b}\right)^2$.

28. $\dfrac{a^2(x - y)}{(a + x)(a + y)^3}$. 29. 1. 30. $\dfrac{1}{x^2}$.

XV.—Page 82.

1. 3. 2. 4. 3. $a + b$. 4. 5.

5. 6. 6. 3. 7. 5. 8. 2.

9. c. 10. $a + b$. 11. $a + b + c$. 12. $\frac{a^2 + ab + b^2}{b}$.

13. abc. 14. $-(a + b)$. 15. abc. 16. $\frac{2a(2b^2 - 5)}{4a - 3b}$

17. 2. 18. 8. 19. $\frac{1}{4}$. 20. $\frac{cd - ab}{a + b - c - d}$.

21. $\frac{a^2 + ab + b^2}{a + b}$. 22. $\frac{1}{abc}$. 23. $a + b + c$.

24. $\frac{1}{a + b + c}$.

XVI.—PAGE 87.

1. 6. 2. $\frac{ab - cd}{a + b - c - d}$. 3. 40.

4. $\frac{3856}{3971}$. 5. $-\frac{31}{159}$. 6. $-\frac{7}{8}$.

7. $1\frac{1}{2}$. 8. 3. 9. $-1\frac{1}{3}$.

10. 8. 11. $1\frac{41}{93}$. 12. $\frac{3 - m - n}{2a}$.

13. $-\frac{a^2 + b^2}{a + b}$. 14. $2\left(\frac{1 - b}{1 + b}\right)^2 - \left(\frac{1 - b}{1 + b}\right)^4$.

15. 3. 16. -8. 17. 1. 18. $\frac{b^2 + 1}{2ab}$.

19. $(\sqrt{a} + \sqrt{b})^2$. 20. $-a$. 21. $a^2 + 2a$.

22. $\frac{6}{7}$. 23. $-2\frac{1}{2}$. 24. $\frac{af(bc - ad) + (cf - ae)cd}{a^2(bc - ad) - (cf - ae)c^2}$.

XVII.—PAGE 92.

1. 10. 2. 45, 25. 3. £360, £240, £120.
4. 20, 15 miles per hour. 5. 24.
6. 36s., 48s. 7. 12. 8. 36s. 9. 30s.
10. 13. 11. 10. 12. 24 lbs., 40 lbs.

13. 13. 14. 500. 15. 15s. 16. $46\frac{6}{19}$.
17. 18 miles from A. 18. $\frac{24}{647}$ hr.
19. 15 per cent. 20. 28 miles. 21. 10.
22. $4\frac{1}{3}$, 4 miles per hour. 23. 12 miles.
24. 20 miles per hour in same direction.
25. 120. 26. 6s., 4s. 27. 2, 1, $\frac{4}{3}$, 6 hours.
28. 98 per cent. 29. £1,000. 30. $38\frac{2}{21}$ miles per hour.
31. 2 inches. 32. 16 vols. of hydrogen, 8 of oxygen.
33. $\frac{1-100c}{b-c}, \frac{100b-1}{b-c}$. 34. $\frac{ab'c - a'bc'}{cc'(a'b - ab')}$.
35. $-40°$. 36. $\frac{a^2b}{c-ab}$.

XVIII.—PAGE 99.

1. 3, 4. 2. 5, 2. 3. 3, 7. 4. 4, 5.
5. 12, 8. 6. 2, 6. 7. 1, 2, 3. 8. 1, 2, 3.
9. 4, 1, 2. 10. $\frac{b_1c - bc_1}{ab_1 - a_1b}, \frac{ac_1 - a_1c}{ab_1 - a_1b}$.
11. $x = \frac{a+c-b}{2}$, &c. 12. $x = \frac{d+f-e}{2a}$.
13. 4, 5. 14. 3, 8. 15. 4, 8. 16. 3, 4.
17. 3, 5. 18. 8, 1. 19. $x = \frac{2}{a+c-b}$, &c.
20. 2, 3, 4, 1. 21. 3, 4, 5. 22. $x = \frac{a+c+d-2b}{3}$, &c.
23. $x = \frac{m(a+c-b)}{2}$, &c. 30. $x = y = z = 1$.

XIX.—PAGE 102.

1. 12, 8. 2. 37. 3. 6 lbs., 5 stones.
4. $\frac{3}{7}$. 5. 17 horses, 24 cows.

6. 6s., 3s. **7.** 220 and £2. 14s.
8. 200, 300, 400. **9.** 48, 23, 18. **10.** 5, 3, $4\frac{1}{2}$.
11. 4, 6, 8 hours. **12.** 6, 10, 18. **13.** $7\frac{1}{2}$, 12, 3.
14. 4, 3 miles per hour. **15.** 16, 32, 48 miles per hour.
16. $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$,
$a_3x + b_3y + c_3z = d_3$, from which x, y, z.
20. $30\frac{9}{20}$, $152\frac{1}{4}$.

XX.—PAGE 109.

1. ± 3. **2.** ± 4. **3.** ± 5.
4. $\pm \frac{2}{3}$. **5.** ± 2. **6.** $\pm (a + b)$.
7. ± 1. **8.** $\pm \sqrt{a^2 - b^2}$. **9.** $\pm \sqrt{2ab - b^2}$.
10. $\pm \frac{2}{3a}\sqrt{3}$. **11.** $\pm a \sqrt{\frac{3b^2 + 1}{b^3 + 3b}}$.
12. $\pm \frac{a + 1}{2\sqrt{a}}$. **13.** 2, 3. **14.** 9, -8.
15. 1, $-\frac{2}{3}$. **16.** 4, $-\frac{7}{4}$. **17.** $-\frac{7}{3}$, $-\frac{5}{2}$.
18. -5, $-\frac{8}{5}$. **19.** $\frac{1}{2}(b \quad \sqrt{b^2 - 4ac})$.
20. 5, $3\frac{1}{2}$. **21.** 7, -6. **22.** 6, $-\frac{2}{3}$.
23. 1, $\frac{ad - (a + b)c}{bc}$. **24.** b, a.
25. $a + b$, $\frac{ab}{a + b}$. **26.** a, b. **27.** $\frac{c}{a - b}$, $\frac{d}{a - b}$.
28. $\frac{a^2 + ab + b^2}{a - b}$, $\frac{a - b}{a^2 - ab + b^2}$. **29.** 4, $\frac{3}{10}$.
30. 5, $-\frac{73}{56}$. **31.** 3, -5. **32.** 2, $-3\frac{1}{2}$.
33. 2, $-\frac{133}{229}$. **34.** 5, $\frac{1}{4}$. **35.** 8, $-\frac{8}{7}$.
36. 3, $-\frac{26}{23}$. **37.** 2, $2\frac{1}{2}$. **38.** 4, $-\frac{5}{3}$.
39. 15, -1. **40.** 2, 0. **41.** 0, $2ab$.

42. $a, 3\frac{2}{5}a$. 43. $0, \frac{a^2 + b^2 + 2b\sqrt{ab}}{(b-a)b}$.

44. $1, \frac{a^2b^2(a^2-b^2)^2}{\{(a^2+b^2)^2 + 2ab(a+b)^2\}^2 - 4a^4b^4}$.

45. $a + b + c, -\frac{a^2+b^2}{a-b}$.

XXI.—Page 115.

1. ± 3. 2. $2, -\sqrt[3]{2\frac{3}{5}}$. 3. $\pm 5, \pm\frac{1}{\sqrt{5}}$.

4. $2, -5$. 5. $\pm\frac{1}{5}$. 6. $\sqrt[n]{2}, \sqrt[n]{3}$. 7. 4, 69.

8. $\pm 4, \pm\sqrt{7}$. 9. $3, -4, 3 \pm \sqrt{21}$. 10. $\frac{8}{3}, 4$.

11. $-2, -2 \pm \sqrt{3}$. 12. $\frac{a}{2}$.

13. $4, \frac{1}{3}, -\frac{1}{6}(-13 \pm \sqrt{145})$. 14. $0, b - a$.

15. $\pm\frac{1}{2}a\sqrt{3}$. 16. $\frac{a}{2} \pm \sqrt{b - 2a\sqrt{b} + \frac{a^2}{2}}$.

17. $0, \left(\frac{a \pm b}{a \mp b}\right)^{\frac{2pq}{p-q}}$. 18. $\pm\frac{a}{\sqrt{2}}\sqrt{\sqrt{5} - 1}$.

19. $\left(\frac{a^{2m} \pm 1}{a^{2m} \mp 1}\right)^2$. 20. $\pm 5, \pm\sqrt{\frac{351}{23}}$.

21. $2, -\frac{1}{3}, \frac{1}{6}(-11 \pm \sqrt{97})$. 22. $4, \frac{3}{8}$.

23. $5, 3\frac{1}{2}, \frac{1}{4}(-7 \pm \sqrt{53})$. 24. $3a$.

25. $3, -2$. 26. $4, -14\frac{1}{2}, \frac{1}{2}(-19 \pm \sqrt{741})$.

27. $3, -2\frac{1}{2}, 1\frac{5}{8}$. 28. $\frac{1}{2}(1 \pm \sqrt{4a - 3})$.

29. ± 1, and $x^4 + x^2 + 1 = 0$. 30. $\pm\sqrt{\frac{a+b+c}{3}}$.

31. $a, \frac{1}{2}\left\{\sqrt{4c^2 - 3a^2 + 2ab + b^2} - (a - b)\right\}$.

32. $x = m$ is one solution.

33. $x = a + b + c$ is one solution.

34. $x = \frac{a}{m}$, and $a^2 + ax + x^2 = 0$.

35. $x = \frac{b + d}{a + c}$ is one solution.

36. $x^2 - x + 1 = 0$ gives two solutions.

XXII.—Page 119.

1. $x = 2, 3, y = 3, 2$. **2.** $x = 5, -3, y = 3, -5$.

3. $x = 3, 4, y = 4, 3$. **4.** $x = 2, 4, y = 4, 3$.

5. $x = 5, -2, y = 2, -5$. **6.** $\pm 7, \pm 6$.

7. $\pm 6, \pm 3$. **8.** $4, 2$. **9.** $\pm 2, \pm 7$.

10. $\pm 4, \pm 3$. **11.** $x = \pm 3, \frac{1}{\sqrt{3}}, y = \pm 2, -\frac{8}{\sqrt{3}}$.

12. $x = 9, 4, y = 4, 9$. **13.** $x = 2, 8, y = 8, 2$.

14. $x = 3, 2, y = 2, 3$.

15. $x = \frac{2}{a \pm \sqrt{2b^2 - a^2}}, \ y = \frac{2}{a \mp \sqrt{2b^2 - a^2}}$.

16. $\pm \frac{(a + 1)b}{\sqrt{2(a^2 + 1)}}, \ \pm \frac{(a - 1)b}{2\sqrt{a^2 + 1}}$.

17. $\pm \sqrt{\frac{a + b^2}{2b}}, \ \pm \sqrt{\frac{a - b^2}{2b}}$.

18. $x = 2, 3, y = 3, 2$. **19.** $x = 2, 3, y = 3, 2$.

20. $x = 2, 3, y = 3, 2$. **21.** $x = 5, -3, y = 3, -5$.

22. $x = \frac{a + \sqrt{2b - a^2}}{\sqrt[3]{4a^2 - 4b}}$. **23.** $x = 2, 3, y = 3, 2$.

24. $x = \pm 3, \mp \frac{15}{8}\sqrt{\frac{32}{35}}, \ y = \pm 2, \pm \sqrt{\frac{32}{35}}$

25. $x = 4, 9, y = 9, 4$. **26.** $x = 4, 1, y = 8, 0$.

27. $x = 4, 3, y = 3, 4$. **28.** $\pm 3, \pm 1$.

29. $x = 0, 3, -\frac{1352}{1145}, \ y = 0, 2, \frac{1768}{1145}$.

30. $0, -1.$ 31. $x = 4, \frac{1708}{559}, y = 3, -\frac{6243}{559}.$

32. $x = 5, 6, y = 4.$ 33. $x = 3, 1, y = 1, 0.$

34. $x = 3, -\frac{138}{45}, y = 1, -\frac{46}{45}.$

35. $x = 2, 7, y = 4, -14.$

36. $x + y = c$, and $xy = bx + ay.$

37. $x = 0, a - b, y = 0, a - b.$

38. $\frac{61}{2} \pm 9\sqrt{\frac{35}{12}}, \frac{1}{2} \pm \sqrt{\frac{35}{12}}.$

39. $x = \sqrt{\dfrac{(a + c - b)(a + b - c)}{2(b + c - a)}}.$

40. $1, 2, 3.$ 41. $\pm 1, 0, 0.$

42. $x = y = z = 0$, also $\dfrac{1}{xy} = \frac{1}{2}\left(\dfrac{1}{b^2} + \dfrac{1}{c^2} - \dfrac{1}{a^2}\right),$

$\dfrac{1}{yz} = \frac{1}{2}\left(\dfrac{1}{a^2} + \dfrac{1}{c^2} - \dfrac{1}{b^2}\right)$, &c.

43. $\dfrac{a}{\sqrt[4]{a^2 + b^2 + c^2}}, \dfrac{b}{\sqrt[4]{a^2 + b^2 + c^2}}, \dfrac{c}{\sqrt[4]{a^2 + b^2 + c^2}}.$

44. $x = y = z = u = a + b.$

45. $\dfrac{a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{\frac{1}{2}}}{(a + b + c)^{\frac{1}{4}}}, \dfrac{a^{\frac{1}{2}}b^{\frac{1}{2}}c^{-\frac{1}{2}}}{(a + b + c)^{\frac{1}{4}}}$, &c.

46. $x = 6, 4, y = 4, 6, z = 5.$

XXIII.—PAGE 123.

1. 9, − 6. 2. Numerator $2\frac{82}{95}$, denominator 3.

3. 35 or 23. 4. 10, − 16. 5. 78, − 68 ft.

6. 2s., 9s. 7. 20 hours. 8. 48 shillings.

9. 3 ft., 2 ft. 10. 1, ½. 11. 3 ft., 4 ft., 5 ft.

12. $\dfrac{\sqrt[n-1]{a_1a_2a_3\ldots a_n}}{a_1}, \dfrac{\sqrt[n-1]{a_1a_2a_3\ldots a_n}}{a_2}$, &c.

13. 2, 3, 6 days.

14. Solve the equations $P(R^3-1)=a$, $PR^2(R-1)=b$, where P is the principal, and R the amount of £1 for one year.

15. Let x be the equated time, then x is the root of the equation—$240\left\{a(m-x)-b(x-n)\right\}=b(x-n)(m-x)$.

16. $x=y=z=\dfrac{1}{\sqrt{2}}$.

17. $\sqrt{ab}$, $\sqrt{ac}$, $\sqrt{bc}$.

XXIV.—Page 129.

1. $\frac{8}{3}\sqrt{165}$. 2. $cx^2+bx+a=0$.

3. $(x+\gamma)^2-p(x+\gamma)+q=0$. 4. 4.

5. $a(ac'-a'c)^2+b(a'b-ab')(ac'-a'c)+c(a'b-ab')^2=0$.

7. $m\pm n\pm p=0$.

8. $x^2-(p+\sqrt{p^2-4q})x+p\sqrt{p^2-4q}=0$.

12. $(3x+4y+5)(2x+3y-2)=0$.

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